

A RESEARCH ON THE SPHERICAL
DYNAMICAL EQUILIBRIUM-
DISTRIBUTION OF STARS
OF UNEQUAL MASSES

BY

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CONTENTS.

	Page.
Chapter I. Introductory historical remarks	5
» II. Equipartition of energy by dynamical equilibrium	12
» III. Spherical distribution of space-coordinates by dynamical equilibrium	23
» IV. Application of theory to the open cluster M 37	42



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CHAPTER I.

Introductory Historical Remarks.

The problems concerning the distribution of the stars in stellar clusters have been studied in two historical essays during the last year.

In the monograph entitled *Sternhaufen*¹⁾ P. TEN BRUGGENCATE gives a fairly comprehensive survey over the historical development of the problems, while in his *Review of Stellar Dynamics* etc.²⁾ J. OHLSSON deals, among other problems regarding steady stellar systems in dynamical equilibrium, with those especially connected with stellar clusters. Referring to the aforementioned essays for a detailed account of all papers on stellar clusters and stellar systems in dynamical equilibrium, I purpose to limit myself to papers bearing specially on the extent a stellar clusters may be analogous to a mass of gas.

The first to point out that the distribution of stars was alike in different stellar clusters was E. C. PICKERING, who, in 1897, published a study on the stellar clusters ω Centauri, 47 Tucanæ and M 13.³⁾ PICKERING proved that the law of apparent distribution of stars — the number of stars per unit of surface in a plan perpendicular to the line of sight, directed to the centre of the cluster, as a function of the apparent distance from the centre, — was identical for all three clusters. He also tried, but without success, to express the law analytically by the formula $(I-r)^n$, r being the distance and n a constant. Nine years later (1906) Professor H. v. ZEIPPEL solved the problem of finding the law of true distribution in space — the number of stars per unit of volume as a function of the angular distance from the centre — the law of apparent distribution being known.⁴⁾ The following year v. ZEIPPEL made the first successful attempt to apply the results obtained from the dynamical theory of gases to the law of distribution of stars in globular clusters⁵⁾.

Many years previously LORD KELVIN had proposed to treat the celestial masses statistically as is done with the molecules of a mass of gas, and G. H. DARWIN

¹⁾ P. t. BRUGGENCATE: *Sternhaufen, ihr Bau, ihre Stellung zum Sternsystem und ihre Bedeutung für die Kosmogonie* (Berlin 1927).

²⁾ J. OHLSSON: *A Review of Stellar Dynamics together with Researches on Steady Stellar Systems in Dynamical Equilibrium* (Lund 1927).

³⁾ E. C. PICKERING: *Distribution of Stars in Clusters*. *Annals of the Astr. Obs. of Harvard College*, Vol. 26, p. 213 (1897).

⁴⁾ H. v. ZEIPPEL: *Catalogue de 1571 étoiles contenues dans l'amas globulaire Messier 3* (N. G. C. 5272). *Ann. Obs. Paris, Mém.* Vol. 25, F (1906).

⁵⁾ H. v. ZEIPPEL: *La théorie des gaz et les amas globulaires*. *Comptes rendus de l'Acad. d. Sciences*, Vol. 144, p. 361 (1907).

had also availed himself of these methods in discussing Sir N. LOCKYER's views on cosmogony.¹⁾

Supposing all the meteorites to have equal masses, DARWIN finds that the law of distribution is expressed by the differential equation that had formerly been given by RITTER,²⁾ HILL³⁾ and THIESEN⁴⁾ (not quoted by DARWIN) for the density of a mass of gas under the influence of its own gravitation. DARWIN, however, also discusses the distribution of unequal masses and gives an expression representing the density of meteorites as a function of the potential of gravitation with respect to the whole system, and an expression for this potential as a function of the density. DARWIN remarks that from these expressions »we should obtain a very complicated differential equation to determine w » (the density of meteorites of a certain mass), »the solution of which is hopelessly difficult.» In order to simplify the operation, DARWIN supposes that the potential is the same one as when all the masses are equal, and he thus obtains an approximative equation for the density.

In 1901—2 LORD KELVIN published two papers in the *Philosophical Magazine*⁵⁾ and, using the methods deducted from the dynamical theory of gas, discussed the distribution of our stellar system in relation to its age. LORD KELVIN, however, did not employ these methods with respect to the distribution of velocities or the distribution of the stars. In a lecture given before the *Société Astronomique de France*, POINCARÉ⁶⁾ (1906) emphasized the great possibilities which would accrue from stellar investigations based on the analogy existing between stellar systems and masses of gas. POINCARÉ further pointed out the difference existing between the two. In a mass of gas the mean free path is small when compared with the dimensions of the gas; but such is not the case in a stellar system. The conditions in a stellar system are best likened to those in a CROOKE's tube where the gas is very rarefied. For this reason it seemed probable to POINCARÉ that a very long time must elapse ere a stellar system, arising out of any distribution, reaches equilibrium. POINCARÉ thought that the distribution of the stars in the Milky Way System is not in equilibrium. In globular clusters, however, the influence of one star on another is more prominent and POINCARÉ suggested that the dis-

¹⁾ G. H. DARWIN: On the mechanical conditions of a swarm of meteorites and on theories of cosmogony. *Phil. Trans. of the Roy. Soc. of London*, Vol. 180 A, p. 1 (1889).

²⁾ A. RITTER: Untersuchungen über die Höhe der Atmosphäre. *Ann. d. Physik u. Chemie*, hrg. v. G. WIEDEMANN, Vol. 16, part 1, p. 166 (1882).

³⁾ G. W. HILL: On the interior constitution of the earth as respects density. *Ann. of Mathematics*, Vol. 4, p. 19 (University of Virginia 1888).

⁴⁾ M. THIESEN: Über die Verbreitung der Atmosphäre. (A. W. SCHADESCHE Buchdruckerei, Berlin 1878).

⁵⁾ LORD KELVIN: On ether and gravitational matter through infinite space. *Philosophical Magazine* Vol. II 6th series, p. 161 (Lecture 16, Baltimore, Oct. 15, 1884). LORD KELVIN: On the Clustering of Gravitational Matter in any part of the Universe. *Philosophical Magazine*, Vol. III 6th series, p. 1.

⁶⁾ H. POINCARÉ: La voie lactée et la théorie des gaz. *Bulletin de la Soc. Astr. de France*, Vol. 25, p. 153 (1906).

tribution of stars in clusters obeys the same law as that followed by the distribution of molecules in a mass of gas exposed to its own gravitation.

POINCARÉ'S suggestion was examined by v. ZEIPEL in his afore-mentioned paper (1907) on *La théorie des gaz et les amas globulaires*. Supposing all stars in a globular clusters to have equal masses, v. ZEIPEL combines the well-known relation of BOLTZMANN between the density and the potential with the relation between the potential and the density, taken from the theory of attraction, and obtains an expression from which the density can be computed for different distances from the centre of the cluster. The comparison established between the computed distribution and the distributions in space, observed in the globular clusters ω Centauri and M 3, shows that the theoretical and observed laws coincide in the central parts of the clusters but do not do so in the outer layers where the theoretically computed densities are found to be too great.

In the same year EMDEN'S monograph, *Gaskugeln*,¹⁾ also appeared and dealt with the problems appertaining to the distributions of spherical masses of gas submitted to their own gravitation. Pressure, density, temperature and potential at a certain place in the mass are connected with one another and with the distance from the centre by three relations, viz: the general equation of state, the hydrostatical equation of equilibrium and the integral-expression giving the potential as a function of the density. Thus there is one supernumerary variable and one more relation required. In a diagram connecting pressure p and specific volume v at different distances from the centre, the points representing the values of pressure and the specific volume can be fixed at random. The relation required is obtained by fixing the points to a certain curve. EMDEN studied gas balls, the curve being determined by the equation:

$$p \cdot v^\gamma = \text{constant}$$

where γ is a constant. Instead of γ another constant n is often used and is connected to γ by the relation

$$\gamma = 1 + \frac{1}{n}$$

When n has any finite value the distribution obtained is called *adiabatic*; when n is infinitely great the distribution is marked by constant temperature and is called *isothermal*.

For $n = \infty$ the density ϱ is determined as a function of the distance r from the centre by the differential equation:

$$\frac{d^2 \log \varrho}{dr^2} + \frac{2}{r} \frac{d \log \varrho}{dr} + a \varrho = 0 \quad . \quad .$$

¹⁾ R. EMDEN: *Gaskugeln, Anwendungen der mechanischen Wärmetheorie auf kosmologische und meteorologische Probleme.* (Leipzig u. Berlin 1907).

for n finite from

$$\frac{d^2 \varrho^{r-1}}{dr^2} + \frac{2}{r} \frac{d \varrho^{r-1}}{dr} + \beta \varrho = 0$$

where α and β are constants. These equations can only be solved in finite terms when $n = 0$, $n = 1$ and $n = 5$. The solutions for $n = 0$ and $n = 1$ cannot be applied to stellar clusters but the solution for $n = 5$ is of great interest. This solution was found by SCHUSTER¹⁾ to be

$$\varrho = \frac{C^{5/2}}{(C^2 + r^2)^{5/2}}$$

where C is a constant.

From the solutions taken when $n = 1$ and $n = 5$, H. C. PLUMMER²⁾ deduced the corresponding distributions of the mass of gas or cluster projected on a plan and compared these distributions with those observed in globular clusters. The solution when $n = 1$ was not suitable. The solution when $n = 5$ fitted well to the outer parts of the clusters but not so well to the central parts. The above-mentioned solution when $n = 5$ is, however, only a particular one found on the supposition that

$$\left(\frac{d \varrho}{dr} \right)_{r=0} = 0$$

Giving up this supposition, v. ZEIPEL elaborated the complete solutions of the differential equation containing two auxiliary constants.³⁾ Applying these solutions to the distributions observed in four clusters by the method of least squares, v. ZEIPEL found that the distributions in the following clusters were satisfied by the following values:

$$\begin{array}{ll} M2 & \gamma = 1.194 \pm 0.014 \\ M3 & \gamma = 1.198 \pm 0.011 \\ M13 & \gamma = 1.203 \pm 0.010 \\ M15 & \gamma = 1.197 \pm 0.024 \end{array}$$

v. ZEIPEL resumed: »Il en faut conclure que pour les amas MESSIER 2, 3, 13 et 15 le nombre n a la valeur $n = 5$.»

In order to explain the value $n = 5$, PLUMMER assumed that the clusters had been formed out of masses of gas having the specific heat ratio of 1.2, the distributions being characterized by $n = 5$. To this hypothesis v. ZEIPEL objected

¹⁾ A. SCHUSTER: On the Internal Constitution of the Sun. Report of the 43rd Meeting of the British Ass. for the Advancement of Science, 1883, p. 427.

²⁾ H. C. PLUMMER: On the problem of distribution in globular star clusters. Monthly Notices of the R. A. S., Vol. 71, p. 460 (1911).

³⁾ H. v. ZEIPEL: Recherches sur la constitution des amas globulaires. Kgl. Svenska Vet. Akad. Handlingar, Vol. 51, No. 5 (1913).

that while the stars were in process of formation they had to move in a resisting medium giving rise to a new distribution of matter which, in itself, was most unlikely to be identical with that found in the mass of gas. v. ZEIPPEL affirmed it quite feasible to compare the stars in a cluster to the molecules in a mass of gas, the interactions between the stars occasioning a certain distribution just as collisions between the molecules also produce a certain distribution.

EDDINGTON¹⁾ has commentated upon v. ZEIPPEL's views. If a globular cluster were a formation analogous to a gas ball it ought to present a law of distribution having $n = \infty$. When the molecules in a gas ball are distributed adiabatically, the distribution is maintained by transports of energy acting conjointly with the transport of kinetic energy of the molecules themselves. But as no similar phenomenon exists in a star cluster, the distribution might well be expected to be analogous to the isothermic one having $n = \infty$. If the distribution nevertheless presents some finite value for n , the theoretical result is $n = 1.5$, just as it is found to be with a mono-atomic gas. This fact had been observed by v. ZEIPPEL himself and, in order to explain the value $n = 5$, he supposed the stars, to a great extent, to be binaries and to possess even still more degrees of freedom. EDDINGTON remarks, however, that it is very incredible that the interior potential energy of a binary system takes part in the equipartition of the energy of the whole cluster, as it ought to do in explanation of the value $n = 5$.

EDDINGTON desists from giving an explanation founded on the theory of specific heats of gases and accentuates the fact, already mentioned by EMDEN, i. e. that when a gas ball is adiabatically distributed and $n = 5$, it extends to infinity and still has a finite mass. It seems very natural to EDDINGTON that the globular clusters should adopt the distribution that gives the greatest extension to their finite mass. This is surely a very remarkable property of the adiabatical distributions when $n = 5$, but as there are other distributions — though not adiabatical (vide BRUGGENCATE, loc. cit. p. 92) — having the same property, EDDINGTON's explanation does not explain why the globular clusters appear to be *adiabatically* distributed. This question is further discussed on p. 41 in this paper and it will be found that, subject to certain conditions, isothermal distributions can also present the property ascribed to the adiabatical distributions when $n = 5$.

Up to 1921 all studies in the distributions of stars in stellar clusters were built on the assumption that all stars had equal masses. In the afore-mentioned paper, given out in 1913, v. ZEIPPEL based this assumption on an observation which showed that stars of different magnitude in the clusters investigated seemed to be equally distributed (though bright stars slightly preponderated in the centre).

In 1921, however, v. ZEIPPEL²⁾ developed a theory with a view to determine

¹⁾ A. S. EDDINGTON: The distribution of stars in globular clusters. Monthly Notices of the R. A. S., Vol. 76, p. 572 (1916).

²⁾ H. v. ZEIPPEL: Die Bestimmung der Massen der Sterne aus ihrer Verteilung in den Sternhaufen, Astr. Nachr. Jubiläums nr. (1921).

the relative mean individual masses of groups of stars whose laws of distribution in globular clusters were observed to be different. Let

$$F(x, y, z, \xi, \eta, \zeta; \mu; t) dx dy dz d\xi d\eta d\zeta d\mu$$

be the number of stars whose coordinates of position are, at the time t , within the limits $x \pm \frac{1}{2} dx$, $y \pm \frac{1}{2} dy$, $z \pm \frac{1}{2} dz$, (Origo in the centre of the cluster), the components of velocities of the stars being within the limits $\xi \pm \frac{1}{2} d\xi$, $\eta \pm \frac{1}{2} d\eta$, $\zeta \pm \frac{1}{2} d\zeta$ and their masses being within the limits $\mu \pm \frac{1}{2} d\mu$, then F is given by the following expression obtained by the method of statistical mechanics according to GIBBS

$$F = \alpha(\mu) e^{-h\mu[\xi^2 + \eta^2 + \zeta^2 - 2V(r)]}$$

where $V(r)$ is the potential of attraction at the point (x, y, z) , h is a constant and $\alpha(\mu)$ a function depending on the relative occurrence of the stars of different sizes. From this expression for F a function $f(r, \mu)$ — easily found by integration — gives the number of stars per unit volume at the distance r from the centre having masses between the limits $\mu \pm \frac{1}{2} d\mu$. We get

$$f(r, \mu) = \beta(\mu) e^{2h\mu V(r)} = \gamma(\mu) [f(r, 1)]^\mu$$

where $\beta(\mu)$ and $\gamma(\mu)$ are functions of $\alpha(\mu)$. Thus

$$\log f(r, \mu) = \log \gamma(\mu) + \mu \log f(r, 1).$$

To determine the two unknown quantities $\log \gamma(\mu)$ and μ just as many equations are obtained as pairs of values of $f(r, \mu)$ and $f(r, 1)$ are observed. These equations are solved by the method of least squares and μ is determined by the arbitrary unit that is chosen.

A paper published the same year by v. ZEIPPEL and J. LINDGREN¹⁾ contained a list of 2.110 stars in the open cluster M 37 ($\alpha = 5^h 46^m$, $\delta = +32^\circ 32'$). Color-indices for 1.885 of these stars were also given. Supposing the mass of a star to be a function of its absolute magnitude and color-index, the 1.885 stars can be classified into groups, according to mass, by dividing the stars into classes of visual magnitude and color-index. As all stars of the cluster are at about the same distance from the observer they are hence reduced from visual magnitude to absolute magnitude by a constant term. The densities of volume of stars of a certain mass $f(r, \mu)$ are then calculated from the observed densities of surface. Now, as v. ZEIPPEL and LINDGREN assume that the stars of different masses are distributed in equilibrium in relation to one another and in relation to surround-

¹⁾ H. v. ZEIPPEL und J. LINDGREN: Photometrische Untersuchung der Sterngruppe Messier 37 (N. G. C. 2099). Kgl. Svenska Vet. Akad. Handlingar, Vol. 61, nr. 15 (1921).

ing stars not belonging to the cluster, it is possible to determine the values of the mean masses of the different groups by the application of the equation between $f(r, \mu)$ and $f(r, I)$. The values of mass found by this method (see Chapter IV) agree with those obtained by other methods, and v. ZEIPEL and LINDGREN conclude from this fact that their assumption with respect to equilibrium was fully borne out.

As, however, the distribution of globular clusters has been found to be not one of isothermal equilibrium, a closer examination of the conclusion of v. ZEIPEL and LINDGREN is of interest. It will be seen in Chapter IV that the observed values of $f(r, \mu)$ (and not only their relations to one another) are in accordance with the values of $f(r, \mu)$ based on the assumption that isothermal equilibrium rules.

The method of v. ZEIPEL and J. LINDGREN has also been exposed and applied to clusters by E. FREUNDLICH and V. HEISKANEN.¹⁾

A closer inspection of the distributions of stars in clusters shows deviations from the spherical shape and distribution (vide BRUGGENCATE) but this paper will assume the distribution to be spherical. This is a first approximation that does not seem to have been studied sufficiently.

I will also neglect all other possible integrals of the motion and only take into account the energy integral, although it has been shown by J. OHLSSON (loc. cit. p. 89) that supplementary integrals permit the giving of a generalised differential equation for the steady states of spherical systems.

¹⁾ E. FREUNDLICH, V. HEISKANEN: Über die Verteilung der Sterne verschiedener Massen in den kugelförmigen Sternhaufen. Zeitschrift für Physik, 14, p. 226 (1923).

E. FREUNDLICH: Zur Dynamik der kugelförmigen Sternhaufen. Physik. Zeitschrift, 24, p. 221 (1923).

CHAPTER II.

Equipartition of Energy by Dynamical Equilibrium.

In order to get a handy expression of the exhaustion of the potential energy of a stellar cluster I will here assume that the effects of collisions and passages are negligible when compared with the effect of the gravitation from the cluster; in short, that the steady state sought is a state of dynamical equilibrium. Further, I will assume that the mean values of kinetic energy per degree of freedom are constant for all degrees of freedom and are independent of the coordinates of position; briefly stated, that there exists equipartition of energy. I intend in this chapter to examine whether these two hypotheses are concordant, and for that reason will prove the principle of equipartition of energy as given by JEANS (vide J. H. JEANS: *The dynamical theory of gases*, 4th Edition, p. 83 et seq.). I will, however, immediately use the word *star* instead of *molecule* and this substitution will be justified by the sequence.

Take a stellar system consisting of n stars. Let the positions of the stars be given in a rectangular system of coordinates by the $3n$ coordinates $q_1, \dots, q_{3n}$. Let $2T$ be the total kinetic energy which is a function of the velocities of the stars $\frac{dq_i}{dt}$. Instead of the velocities $\frac{dq_i}{dt}$, introduce new variables p_i defined by

$$\frac{\partial T}{\partial \frac{dq_i}{dt}} = p_i \quad (i = 1, \dots, 3n).$$

Let us suppose that the forces acting on the stars have a potential function U , which is a function of the coordinates $q_1, \dots, q_{3n}$, but independent of the velocities. The motion is then determined by the following equations

$$\left. \begin{aligned} \frac{dq_i}{dt} &= \frac{\partial E}{\partial p_i} \\ \frac{dp_i}{dt} &= - \frac{\partial E}{\partial q_i} \end{aligned} \right\} \quad (i = 1, \dots, 3n)$$

where

$$E = T - U$$

Let us, however, make the more general assumption that E is no constant but variable with the time

$$\frac{dE}{dt} = -2F.$$

Then the equations of motion are

$$(II, 1) \quad \left. \begin{aligned} \frac{dq_i}{dt} &= \frac{\partial E}{\partial p_i} \\ \frac{dp_i}{dt} &= -\frac{\partial E}{\partial q_i} - \frac{\partial F}{\partial \frac{dq_i}{dt}} \end{aligned} \right\} \quad (i = 1, \dots, 3n).$$

Now consider a space having $6n$ dimensions and let the position of a point in this space be determined by $6n$ rectangular coordinates. Giving these coordinates the values $q_1, \dots, q_{3n}, p_1, \dots, p_{3n}$ a correspondence between the stellar system and the $6n$ dimensional space ensues in such a way that the stellar system is represented at every moment by one single point in the $6n$ -dimensional space. The motion of this single point is determined by the equations (II, 1).

The life history of the stellar system is represented by a certain curve in the $6n$ -dimensional space. Now let us start a great number of stellar systems and suppose that the initial values of the $6n$ quantities $q_1, \dots, q_{3n}, p_1, \dots, p_{3n}$ of the different systems are chosen so near to each other that the representative points can be treated as forming a continuous medium. Let τ be the density of this medium ($=$ number of points per unit of $6n$ -dimensional volume). Then τ is a function of the $6n$ quantities $q_1, \dots, q_{3n}, p_1, \dots, p_{3n}$ and of the time t . τ must, however, satisfy the following equation of continuity

$$(II, 2) \quad \frac{\partial \tau}{\partial t} + \sum_{i=1}^{3n} \left\{ \frac{\partial}{\partial p_i} (\tau \dot{p}_i) + \frac{\partial}{\partial q_i} (\tau \dot{q}_i) \right\} = 0.$$

Introducing the symbol

$$\frac{D\tau}{Dt} = \frac{\partial \tau}{\partial t} + \sum_{i=1}^{3n} \left\{ \frac{\partial \tau}{\partial p_i} \dot{p}_i + \frac{\partial \tau}{\partial q_i} \dot{q}_i \right\}$$

the equation of continuity becomes

$$(II, 3) \quad \frac{D\tau}{Dt} + \tau \sum_{i=1}^{3n} \left\{ \frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} \right\} = 0.$$

But from (II, 1) we have

$$\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} = -\frac{\partial^2 F}{\partial p_i \partial \dot{q}_i} \quad (i = 1, \dots, 3n).$$

and thus

$$(II, 4) \quad \frac{D\tau}{Dt} = \tau \sum_{i=1}^{3n} \frac{\partial^2 F}{\partial p_i \partial \dot{q}_i}$$

It can now be shown (vide JEANS, loc. cit. p. 71) that the right membrum of this equation is always ≥ 0 , and only $= 0$ if $F = 0$.

Hence we get an important result relating to stellar systems. As a consequence of the collisions between stars mechanical energy is transformed into heat and

$$\frac{dE}{dt} = -2F \neq 0.$$

These equations show that the points representing different initial configurations of stellar systems are crowded together and that, independently of the initial states, the final states of the stellar systems are the same. The evolution leading to the final state has been studied by means of other methods by C. F. LUNDAHL.¹⁾

The effect of collisions can, however, be fully neglected. J. OHLSSON (loc. cit. pp. 60 and 62) estimates the time of relaxation re evolution, as an effect of collisions leading to equilibrium, to be of the order

$$10^{21} \text{ years}$$

while the time of relaxation re evolution as an effect of passages is found to be of the order

$$10^{15} \text{ years}$$

when the stellar system is the Galaxy. These quantities are certainly much smaller in globular and open clusters but as in all cases the passages are purposely neglected in this paper, it is evident that the collisions are still more to be disregarded.

Then we have

$$F = 0$$

and the equation (II, 4) is reduced to

$$(II, 5) \quad \frac{D\tau}{Dt} = 0$$

which expresses the theorem of LIOUVILLE.

If the initial values of the stellar systems are chosen in such a manner that the representative points in the $6n$ -dimensional space form a homogeneous medium, the density of this medium will remain constant and the systems will not evolve towards any final state. If the systems are found to always possess a certain property, all the representative points possess this property. Assuming that the path of any point is homogeneously filling the allotted space (postulate of MAXWELL, formulated by POINCARÉ²⁾), we can define the probability that a system

¹⁾ C. F. LUNDAHL: Contribution to statistical mechanics based on the law of Newton. Kungl. Fysiograf. Sällsk. Handl. N. F. Vol. 38 No. 3. (Medd. fr. Lunds Astron. Obs., Ser. II, Nr. 45, Festschrift tillägnad C. V. L. CHARLIER, Lund 1927).

²⁾ H. POINCARÉ: Les hypothèses cosmogoniques, 2^e éd., p. 101, Paris 1913.

has of possessing a certain property as being proportional to the space occupied by representative points possessing the property.

Let us now consider a part of the stellar system consisting of N stars. Suppose that the stars have different masses but can be ordered into ν groups, all the stars in a group having the same mass μ_j ($j = 1, \dots, \nu$). Let N_j be the number of stars having the mass μ_j . Then

$$(II, 6) \quad \sum_{j=1}^{\nu} N_j = N.$$

Assume the radius of every star to be small enough to be neglected; that is, assume the stars to be »material points». Every star then possesses three degrees of freedom and its state is determined by 6 quantities

$$\varphi_1, \varphi_2, \varphi_3, \varphi_4, \varphi_5, \varphi_6.$$

The position and motion of a star in a 6-dimensional space can be represented by a single point and its path. In this space every point can be chosen independently of all the others, the points having no volumes.

The 6-dimensional space that contains N representative points is divided into n equal elements of volume. (Observe the new meaning attached to n). The state of the stars is then statistically determined by quantities a_{ij} giving the number of stars of mass μ_j contained in the i -th element of volume. These numbers must, of course, satisfy the equations

$$(II, 7) \quad \sum_{i=1}^n a_{ij} = N_j \quad (j = 1, \dots, \nu).$$

In accordance with JEANS' nomenclature we shall call the determined distribution of the N stars A . Similarly there exists another distribution B , determined by numbers b_{ij} , and so on. Our task is to determine the a_{ij} 's in such a manner that the distribution A becomes the most probable one. Now, consider the corresponding 6 N -dimensional space. Suppose it possible to numerate all the stars and that every star has a determined position. Then the state of the system is represented by one point in the 6 N -dimensional space. Assuming the coordinates of the stars to be varied between certain limits in such a manner that every star is still within its element of volume, the representative points in the 6 N -dimensional space will fill a certain volume of the said space. Let ω^N represent this element of volume of the 6 N -dimensional space. The statistically determined distribution A is also obtained if two stars of the same mass are interchanged. But then the representative point in the 6 N -dimensional volume is moved into another element of volume. The N_j stars of mass μ_j can be permuted in

$$\frac{N_j!}{a_{1j}! a_{2j}! \dots a_{nj}!}$$

ways and the distribution is still the distribution A . The volume W_A in the $6N$ -dimensional space that contains representative points giving distribution A is thus found to be

$$(II, 8) \quad W_A = \prod_{j=1}^{\nu} \frac{N_j!}{a_{1j}! a_{2j}! \dots a_{nj}!} \omega^N$$

The distribution A is the most probable one if the a_{ij} 's are determined in such a manner that W_A is a maximum. Instead of W_A we need only study the factor

$$\Theta_A = \prod_{j=1}^{\nu} \frac{N_j!}{a_{1j}! a_{2j}! \dots a_{nj}!}$$

We have

$$\log \Theta_A = \sum_{j=1}^{\nu} \log N_j! - \sum_{j=1}^{\nu} \sum_{i=1}^n \log a_{ij}!$$

and, supposing the a_{ij} 's to be so great that they may be treated as infinite, we apply the formula of STIRLING

$$(II, 9) \quad L^t \log p! = \frac{1}{2} \log 2\pi + (p + \frac{1}{2}) \log p - p$$

$$p = \infty$$

and obtain

$$\begin{aligned} \log \Theta_A = & \sum_{j=1}^{\nu} \left\{ \frac{1}{2} \log 2\pi + (N_j + \frac{1}{2}) \log N_j - N_j - \right. \\ & \left. - \sum_{i=1}^n \left[\frac{1}{2} \log 2\pi + (a_{ij} + \frac{1}{2}) \log a_{ij} - a_{ij} \right] \right\} \end{aligned}$$

or

$$\Theta_A = \prod_{j=1}^{\nu} n^{N_j + \frac{1}{2}} (2\pi N_j)^{-\frac{n-1}{2}} e^{-N_j} K_{Aj}$$

where we have introduced

$$K_{Aj} = \frac{1}{N_j} \sum_{i=1}^n (a_{ij} + \frac{1}{2}) \log \frac{n a_{ij}}{N_j}.$$

Θ_A is a maximum if

$$\sum_{j=1}^{\nu} K_{Aj}$$

is a minimum. In order to find this minimum we will treat the K_{Aj} 's as continuous functions of the a_{ij} 's, and by variation we find the condition

$$(II, 10) \quad 0 = \delta \sum_{j=1}^{\nu} K_{Aj} = \sum_{j=1}^{\nu} \sum_{i=1}^n \left\{ \log \frac{n a_{ij}}{N_j} + 1 + \frac{1}{2 a_{ij}} \right\} \delta a_{ij}.$$

The variations δa_{ij} are, however, not independent of one another but have still to satisfy $\nu + 1$ conditions. The ν conditions are found by variations of the equations (II, 7) and are

$$(II, 11) \quad \sum_{i=1}^n \delta a_{ij} = 0 \quad (j = 1, \dots, \nu)$$

The remaining equation is obtained from the following condition expressing the fact that the total energy must rest constant

$$(II, 12) \quad E = \text{constant}.$$

Taking

$$(II, 13) \quad \frac{\partial E}{\partial a_{ij}} = \varepsilon_{ij}$$

and disregarding terms of higher order of δa_{ij} we obtain by variation from the equation (II, 12)

$$(II, 14) \quad \sum_{j=1}^{\nu} \sum_{i=1}^n \varepsilon_{ij} \delta a_{ij} = 0.$$

We are now able to find the equation that is to be satisfied by the a_{ij} 's in order that the distribution A shall possess an extremum of probability. Multiply the equations (II, 11) by the arbitrary factors l_j , the equation (II, 14) by a factor m and finally add the expressions obtained to (II, 10). We thus obtain

$$(II, 15) \quad 0 = \sum_{j=1}^{\nu} \sum_{i=1}^n \left\{ \log \frac{n a_{ij}}{N_j} + 1 + \frac{1}{2 a_{ij}} + l_j + m \varepsilon_{ij} \right\} \delta a_{ij}$$

and the a_{ij} 's are to satisfy the equations

$$(II, 16) \quad 0 = \log \frac{n a_{ij}}{N_j} + 1 + \frac{1}{2 a_{ij}} + l_j + m \varepsilon_{ij}.$$

The mathematical treatment so far permits the a_{ij} 's to give a distribution having either maximum or minimum of probability. But further examination, not reproduced here, has shown that the calculation really leads to a maximum.

We have already supposed the a_{ij} 's to be so great that they may be treated as being infinite and the term $\frac{1}{2 a_{ij}}$ may accordingly be neglected. The most probable stationary distribution of stars having masses μ_i is thus determined by the equations

$$(II, 17) \quad 0 = \log \frac{n a_{i1}}{N_1} + 1 + l_1 + m \varepsilon_{i1}.$$

Let us now assume that ε_{iI} is independent of a_{iI} , though dependent of other a_{ij} 's. Then the equation is easily solved into

$$a_{iI} = \frac{N_I}{n} e^{-(I + l_I) - m \varepsilon_{iI}}$$

Introduce a function τ_j in such a manner that

$$\tau_j d\varphi_{j1} d\varphi_{j2} d\varphi_{j3} d\varphi_{j4} d\varphi_{j5} d\varphi_{j6}$$

gives the number of stars whose φ_k 's satisfy

$$\varphi_{jk} - \frac{1}{2} d\varphi_{jk} < \varphi_{jk} < \varphi_{jk} + \frac{1}{2} d\varphi_{jk} \quad (k = 1, \dots, 6)$$

and whose masses are μ_j .

Obviously τ_I is proportional to a_{iI} and we have

$$\tau_I = C_I e^{-m \varepsilon_I}$$

where C_I is a constant and ε_I represents a smooth function for the values ε_{iI} .

Assume ε_I to be of the form

$$\varepsilon_I = \frac{1}{2} \beta_{11} \varphi_{11}^2 + \frac{1}{2} \beta_{12} \varphi_{12}^2 + \beta_{13} \varphi_{13}^2 + f_1(\varphi_{14}, \varphi_{15}, \varphi_{16}).$$

Then we get

$$\begin{aligned} \tau_I d\varphi_{11} d\varphi_{12} d\varphi_{13} d\varphi_{14} d\varphi_{15} d\varphi_{16} = C_I & \left(e^{-\frac{1}{2} m \beta_{11} \varphi_{11}^2} d\varphi_{11} \right) \dots \\ & \dots \left(e^{-m f_1} d\varphi_{14}, d\varphi_{15}, d\varphi_{16} \right) \end{aligned}$$

an equation showing that the distributions of the coordinates φ_{11} , φ_{12} and φ_{13} are independent of one another.

The mean value of the contribution of φ_{11} to the energy is

$$\overline{\frac{1}{2} \beta_{11} \varphi_{11}^2} = \frac{1}{2m}$$

and we have similar expressions for φ_{12} and φ_{13}

$$\overline{\frac{1}{2} \beta_{12} \varphi_{12}^2} = \overline{\frac{1}{2} \beta_{13} \varphi_{13}^2} = \frac{1}{2m}.$$

For another group of stars having masses μ_j we have in the same manner

$$\tau_j = C_j e^{-m \varepsilon_j}$$

and supposing

$$\varepsilon_j = \frac{1}{2} \beta_{j1} \varphi_{j1}^2 + \frac{1}{2} \beta_{j2} \varphi_{j2}^2 + \frac{1}{2} \beta_{j3} \varphi_{j3}^2 + f_j(\varphi_{j4}, \varphi_{j5}, \varphi_{j6})$$

we find

$$\overline{\frac{1}{2} \beta_{j1} \varphi_{j1}^2} = \overline{\frac{1}{2} \beta_{j2} \varphi_{j2}^2} = \overline{\frac{1}{2} \beta_{j3} \varphi_{j3}^2} = \frac{1}{2m}.$$

We now identify the quantities φ as canonical p -coordinates and find

$$(II, 18) \quad \frac{1}{2} \mu_j \left(\frac{d q_{jk}}{dt} \right)^2 = \frac{1}{2m} \quad (k = 1, 2, 3; j = 1, \dots, v)$$

— the law of equipartition of energy.

When the method of statistical mechanics is applied to stellar systems one of the assumptions, viz. that of ε_{ij} being independent of a_{ij} , requires an especial examination.

Professor CHARLIER¹⁾ and JEANS²⁾ based their statistical-mechanical studies of stellar systems on BOLTZMANN'S equation. Let

$$F(x, y, z, u, v, w, t) \, dx \, dy \, dz \, du \, dv \, dw$$

design the number of stars that at the time t have their coordinates of position within the limits

$$\begin{aligned} x &\pm \frac{1}{2} dx \\ y &\pm \frac{1}{2} dy \\ z &\pm \frac{1}{2} dz \end{aligned}$$

and their velocities within the limits

$$\begin{aligned} u &\pm \frac{1}{2} du \\ v &\pm \frac{1}{2} dv \\ w &\pm \frac{1}{2} dw \end{aligned}$$

then the distribution must always satisfy the equation

$$(II, 19) \quad \frac{dF}{dt} = \nabla(F) + \square(F)$$

where $\frac{dF}{dt}$ stands for

$$\frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} + \frac{\partial V}{\partial x} \frac{\partial F}{\partial u} + \frac{\partial V}{\partial y} \frac{\partial F}{\partial v} + \frac{\partial V}{\partial z} \frac{\partial F}{\partial w}$$

¹⁾ C. V. L. CHARLIER: Statistical mechanics based on the law of Newton. Kungl. Fysiograf. Sällsk. Handl. N. F. Vol. 28. Nr. 5 (Medd. fr. Lunds Astron. Obs. Ser. II, N:o 16) (Lund 1917).

²⁾ J. H. JEANS: On the theory of star-streaming and the structure of the universe, Monthly Notices of the R. A. S. Vol. 76 p. 70, 552 (1915—16). J. H. JEANS: Problems of cosmogony and stellar dynamics (Cambridge 1919).

V is the potential of the gravitational forces; $\nabla (F)$ is a function expressing the variation of F depending on the passages between the stars and $\square (F)$ expresses the variation of F depending on the collisions. As already observed (page 14), the function $\square (F)$ may be neglected when compared with $\nabla (F)$.

The equation
$$\frac{d F}{d t} = \nabla (F)$$

has been thoroughly studied by CHARLIER (loc. cit.) and J. OHLSSON (loc. cit. page 22), who gave the solution in an A type-series taken from mathematical statistics. Terms up to and including moments of the fourth order were calculated and CHARLIER found a solution that asymptotically gave the law of MAXWELL when t increases to infinity. This result, obtained as a solution of the equation

$$\frac{d F}{d t} = \nabla (F)$$

combined with

$$\frac{\partial F}{\partial t} = 0$$

gives a state of equilibrium that CHARLIER¹⁾ calls *statistical equilibrium*. Thus the statistical equilibrium is the final state of the development of a stellar system whose individuals move under the influence of both the gravitation exercised by the other individuals taken as whole (regular forces) and the gravitational forces arising out of the chance approaches of neighbouring individuals (irregular forces). The material collisions are the only effects disregarded. If the irregular forces are also disregarded the equilibrium state is determined by the equations

$$(II, 20) \quad \frac{d F}{d t} = 0, \quad \frac{\partial F}{\partial t} = 0$$

The equilibrium state, determined by these equations, is termed by CHARLIER *dynamical equilibrium*. JEANS (loc. cit.) has propounded a very elegant method for solving these equations and the solutions have been further studied by CHARLIER (independently of JEANS) and by J. OHLSSON (loc. cit.).

The two equation (II, 20) may be condensed into the following equation

$$(II, 21) \quad u \frac{\partial F}{\partial x} + v \frac{\partial F}{\partial y} + w \frac{\partial F}{\partial z} + \frac{\partial V}{\partial x} \frac{\partial F}{\partial u} + \frac{\partial V}{\partial y} \frac{\partial F}{\partial v} + \frac{\partial V}{\partial z} \frac{\partial F}{\partial w} = 0.$$

If the masses of the stars are not all alike but belong to several different classes,

¹⁾ C. V. L. CHARLIER: Über hydrodynamisches Gleichgewicht in Sternsystemen. Ark. f. mat., astr. o. fysik utg. av Kungl. Sv. Vet. Akad. Vol. 12. Nr. 21. (Medd. fr. Lunds Astr. Obs. Ser. I No 82, 1917) C. V. L. CHARLIER: Conceptions monistique et dualistique de l'univers stellaire. Scientia Vol 22. p. 77 (Bologna). (Medd. fr. Lunds Astr. Obs. Ser. I No 81, 1917).

similar equations for the distributions of all classes necessarily exist and all equations are *identically equal*. It thus appears that this method gives no possibility of finding laws connecting the distributions of the stars belonging to different classes of mass. It is to be observed that the equation (II, 21) does not require equipartition of energy.

The method of statistical mechanics, as taken from JEANS' treatise and exposed in this chapter, leads to equipartition of energy and I will therefore here discuss the extent one may be permitted to apply it to the study of stellar systems. All assumptions made, with the exception of two, can just as well be accepted in relation to stellar systems as to gaseous masses. One of these exceptions is that the stellar system, contrary to the gaseous mass, is not enclosed between walls. That question will, however, be discussed in the next chapter. The other exception is that in a stellar system where the irregular forces are neglected, nothing parallel is found to the collisions between the molecules of the gas. This latter difference has bearings on the assumption that ε_{ij} is independent of a_{ij} .

In the first place we observe that the part of E standing for kinetic energy gives terms contributing to the ε_{ij} , and that these terms are independent of a_{ij} . As for the potential energy we will first take into account both regular and irregular forces. The potential energy is then made up of terms each containing the product of two masses divided by their distance. A certain mass in a certain cell is thus multiplied by all other masses, including those of the same cell. And such is the case with all masses of the cell in question. If the potential energy could be expressed as a function of the number of masses in each cell we should find the potential energy to be a function of a higher order than the first of this number. On varying these numbers, the factors ε_{ij} , by which the variations δa_{ij} are multiplied, would still be functions of a_{ij} and our assumption would not be fulfilled. This reasoning is, strictly speaking, also valid for the infinitely small hard elastic molecules of a gas, but in the theory of gas it is assumed that the molecules acting at any time on any of the others are infinitely few and that E is thus solely made up of terms representing kinetic energy.

Now let the irregular forces be disregarded. The potential energy is accordingly calculated by supposing all the masses in a cell to be concentrated at the centre of the cell, and the potential energy of the interactions of the stars in every cell is neglected. Then, expressing the potential energy as a function of the a_{ij} 's, we obtain terms containing the products

$$a_{ij} a_{kl} \quad \begin{matrix} k = i \\ l = j \end{matrix}$$

but the terms

$$a_{ij}^2$$

are vanishing and ε_{ij} is independent of a_{ij} but is function of the a_{kl} ($k \neq i, l \neq j$). Thus we find that the above deductions are applicable to the stellar systems and that equipartition of energy exists in a state of dynamical equilibrium.

At first sight this seems contradictory to the result obtained by the method based on BOLTZMANN'S equation. But such is not the case. The equation (II, 21) is satisfied by all stationary distributions, whilst the method founded on LIOUVILLE'S theorem leads to the most probable distribution. It is thus very likely that the equipartition of energy is a property that does not belong to all stationary distributions but only to the most probable one.

Huge difficulties seem to beset us in our search for the mechanism resulting in the equipartition of energy when collisions and passages are neglected. For the present I will content myself with quoting the following postulate formulated by CHARLIER:¹⁾ »Each population of individuals approaches asymptotically that state which can be proved to possess the greatest probability.«

¹⁾ C. V. L. CHARLIER: On the structure of the universe. Publ. of the Astron. Soc. of the Pacific, Vol. 37, p. 125 (1925).

CHAPTER III.

Spherical Distribution of Space-Coordinates by Dynamical Equilibrium.

In the previous chapter we deduced the equation

$$\tau_j = C_j e^{-m \varepsilon_j}$$

from which v. ZEIPEL and J. LINDGREN found, by integration over all values of the velocities, certain properties relating to the laws of distribution of stars of different masses. In this chapter I will connect the deduction giving the laws of distribution of positional coordinates to the theorem of the viriel of CLAUSIUS, which theorem, as pointed out by POINCARÉ¹⁾ and EDDINGTON,²⁾ has bearings on stellar systems. I will only take one of the results come to in the previous chapter, namely, that the equipartition of energy exists when a stellar system is in its most probable state, even if passages are neglected.

When using GIBBS³⁾ terminology the theorem of LIOUVILLE shows that the *extension in phase* is constant. The *extension in phase* being the product of the *extension in space* and the *extension in velocity*, and the latter being constant, as seen in the previous chapter, we find that the *extension in space* is constant. Hence we may use a $3N$ -dimensional space of the $3N$ positional coordinates serving as a basis of probability, just as we made use of the $6N$ -dimensional space in the previous chapter. In order to find the most probable distribution in space we shall use the method we employed when determining the most probable distribution of velocities. We shall also use the same symbols but attach to each of them a new meaning.

Let us take N stars within a limited volume. The volume may be divided into n equal elements and the distribution A is then determined by quantities a_{ij} , each a_{ij} giving the number of stars having mass μ_j and situated in the element i . Let N_j represent the total number of stars having the mass μ_j and we have

$$(III, 1) \quad \sum_{i=1}^n a_{ij} = N_j \quad (j = 1, \dots, r)$$

As in the foregoing chapter, the most probable distribution of the positional coordinates is determined by the equations

¹⁾ H. POINCARÉ: Les hypothèses cosmogoniques, 2^e éd. p. 93. (Paris 1913).

²⁾ A. S. EDDINGTON: The kinetic energy of a star cluster, Monthly Notices of the R. A. S. Vol. 76 p. 525.

³⁾ W. GIBBS: Elementary principles of statistical mechanics. (New Haven 1902).

$$(III, 2) \quad 0 = \sum_{j=1}^{\nu} \sum_{i=1}^n \left\{ \log \frac{n a_{ij}}{N_j} + 1 + \frac{1}{2} \frac{1}{a_{ij}} \right\} \delta a_{ij}$$

$$(III, 3) \quad 0 = \sum_{i=1}^n \delta a_{ij} \quad (j = 1, \dots, \nu).$$

If these equations were solved we should find a_{ij} independent of i , that is to say, the stars would be uniformly distributed over the whole space. We are not, however, to seek for the most probable distribution among *all* possible ones, but only among those having a certain value of their potential energy, as is made obvious by the theorem of the viriel.

Let q_k be the rectangular coordinates of the N stars of the system along one of the directions of coordinates ($k = 1, \dots, N$). If Q_k are the component of the acting forces in the same direction we get the equations

$$(III, 4) \quad \mu_k \frac{d^2 q_k}{dt^2} = Q_k \quad (k = 1, \dots, N)$$

Similar equations are valid for the coordinates along the other two directions of coordinates. Identically we have

$$\frac{1}{4} \frac{d^2}{dt^2} q_k^2 = \frac{1}{2} \left(\frac{d q_k}{dt} \right)^2 + q_k \frac{d^2 q_k}{dt^2}$$

and thus

$$\frac{1}{4} \mu_k \frac{d^2}{dt^2} q_k^2 = \frac{1}{2} \mu_k \left(\frac{d q_k}{dt} \right)^2 + \frac{1}{2} q_k Q_k$$

Let the summation over all three directions of coordinates be designated by S and we obtain by summation over all stars and all three directions

$$(III, 5) \quad \frac{1}{4} S \sum_{k=1}^N \mu_k \frac{d^2}{dt^2} q_k^2 = \frac{1}{2} S \sum_{k=1}^N \mu_k \left(\frac{d q_k}{dt} \right)^2 + \frac{1}{2} S \sum_{k=1}^N q_k Q_k$$

Now

$$S \sum_{k=1}^N \mu_k q_k^2$$

is the moment of inertia of the system and, as we are seeking stationary distributions, we write

$$\frac{d^2}{dt^2} S \sum_{k=1}^N \mu_k q_k^2 = S \sum_{k=1}^N \mu_k \frac{d^2}{dt^2} q_k^2 = 0$$

the bar over an expression denoting an average over a long time. Further

$$\frac{1}{2} S \sum_{k=1}^N \mu_k \left(\frac{d q_k}{dt} \right)^2$$

is the kinetic energy of the system and is represented by T . Introducing the symbol

$$V = - \frac{1}{2} S \sum_{k=1}^N q_k Q_k$$

we obtain from the equation (III,5)

$$(III, 6) \quad \theta = \overline{T} - \overline{V}.$$

V is the viriel of Clausius.

Assuming that the equation is also valid at every moment we get

$$(III, 7) \quad \theta = T - V.$$

Before applying this equation to the stellar system I will base my further deduction on the assumption that the system is spherically distributed. Among all the stars of the system we will take into consideration N stars that are situated within a sphere of radius R having its centre in the centre of the spherical distribution. All the equal elements of volume that form the basis of the description of the distribution by the quantities a_{ij} are chosen in spherical shells.

The forces Q_k acting on the stars consist of the gravitational forces of stars within and outside the imaginary limiting sphere. As we suppose, however, that the system possesses spherical symmetry and that every star is acted on by the *regular* gravitational forces of other stars and not by forces exercised by every neighbouring star, the total gravitational force of all stars outside the limiting sphere is *nil* and we have only to take into account the forces due to the stars inside the sphere. We must, however, consider the fact that stars inside the sphere at a certain moment may, while following their motions, escape from the sphere. These losses are counterbalanced by outside stars passing into the sphere, and as it is supposed that the distribution is stationary both outside and inside the limiting sphere, the counterbalancing effect appears statistically to be just as if the outgoing stars in the imaginary limiting sphere met an elastic wall and were reflected by it. The stellar system within the limiting sphere can thus be treated as a stationary system, just as is done with the molecules of a gas enclosed by fixed walls. There is, however, one difference, namely, that the stars within the limiting sphere do not maintain their identity after their imaginary reflection; it is a new star that pursues the broken orbit. — Formally, we are able to take account of this fact by introducing a fictitious field of force extending inward from the imaginary sphere in such a manner that every star piercing this field gets its motion

changed according to the laws of impacts. In short, we change the real stellar system into a fictitious one enclosed by elastic walls. The expression of potential energy is not changed except in the immediate neighbourhood of the limiting sphere and this is also the case with the law of distribution, which is a function of the potential energy.

We thus assume that N stars move under the sole influence of their gravitational interactions. It follows that a simple relation exists between the viriel, V , and the exhaustion of potential energy of these forces. The exhaustion of potential energy, Ω , is

$$\Omega = \frac{1}{2} \sum_{k=1}^N \sum_{l=1}^N \kappa \mu_k \mu_l \frac{1}{[S (q_l - q_k)^2]^{1/2}}$$

where the apostrophe by the Σ sign means that during the summation $k = l$ must be excluded, and κ is a numerical constant whose value depends on the units which are to be settled later on.

In the expression

$$V = -\frac{1}{2} S \sum_{k=1}^N q_k Q_k$$

we divide Q_k into terms, each term expressing the force between the star of index k and another one.

$$Q_k = \sum_{l=1}^N Q_{kl}$$

The force expressing the attraction on the mass μ_1 by μ_2 is

$$Q_{12} = \kappa \mu_1 \mu_2 \frac{q_2 - q_1}{[S (q_2 - q_1)^2]^{3/2}}$$

and the attraction of μ_1 on μ_2 is

$$Q_{21} = \kappa \mu_1 \mu_2 \frac{q_1 - q_2}{[S (q_2 - q_1)^2]^{3/2}}$$

On taking the sum over the three directions of coordinates we find

$$S (q_1 Q_{12} + q_2 Q_{21}) = -\kappa \mu_1 \mu_2 \frac{1}{[S (q_2 - q_1)^2]^{1/2}}.$$

From this and similar expressions we obtain (vide EDDINGTON, loc. cit.)

$$V = -\frac{1}{2} S \sum_{k=1}^N q_k \sum_{l=1}^N Q_{kl} = \frac{1}{4} \sum_{k=1}^N \sum_{l=1}^N \kappa \mu_k \mu_l \frac{1}{[S (q_2 - q_1)^2]^{1/2}} = \frac{1}{2} \Omega.$$

The equation (III, 7) thus gives

$$(III, 8) \quad \theta = T - \frac{1}{2} \Omega$$

But the equations of motion (III, 3) possess an integral

$$(III, 9) \quad -\frac{1}{2} h = T - \Omega$$

where h is an arbitrary constant. From (III, 8) and (III, 9) we finally obtain

$$(III, 10) \quad \Omega = h.$$

Just as in the previous chapter the condition

$$E = \text{constant}$$

limited the number of distributions, among which the most probable one was to be chosen, so we will here require the distributions to satisfy the condition (III, 10).

This condition is to be expressed by a relation between the variations δa_{ij} and our first task is thus to express Ω as a function of the quantities a_{ij} .

The exhaustion of potential energy consists of a sum of terms, each term representing the attraction between two stars and each pair of stars being taken once. We arrange all these pairs in a manner that enables us to combine every star with all other stars situated nearer to the centre of the distribution than the star taken. By following this rule, when combining the stars into pairs, every possible pair is obtained once and only once. If the indices k are chosen in a manner that

$$S q_k^2 < S q_{k+1}^2$$

the exhaustion of potential energy may be expressed by

$$\Omega = \sum_{k=1}^N \sum_{l=1}^{k-1} \propto \mu_k \mu_l \frac{1}{[S (q_l - q_k)^2]^{1/2}}$$

Our assumption to neglect irregular forces permits us to substitute for the attraction of a number of discrete stars the attraction of a homogeneous sphere having the same mass as the sum of the masses of the discrete stars. The terms expressing the exhaustion of potential energy of attractions on the mass μ_s

$$\propto \mu_s \sum_{l=1}^{s-1} \frac{\mu_l}{[S (q_l - q_s)^2]^{1/2}}$$

are thus substituted by the expression

$$\propto \mu_s \sum_{l=1}^{s-1} \frac{\mu_l}{[S q_s^2]^{1/2}}$$

Now let us consider the distribution A determined by the numbers a_{ij} . All stars having the same index i are situated within a spherical shell. Let the harmonic mean distance of these stars from the centre be r_s . Then the stars in the element s contribute to the exhaustion of potential energy by a quantity

$$\kappa \frac{\sum_{j=1}^{\nu} a_{sj} \mu_j}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j$$

and we obtain

$$\Omega = \kappa \sum_{s=1}^n \frac{\sum_{j=1}^{\nu} a_{sj} \mu_j}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j.$$

The equation (III, 10) then gives the following relation between the quantities a_{ij} or a_{sj}

$$(III, 11) \quad h = \kappa \sum_{s=1}^n \frac{\sum_{j=1}^{\nu} a_{sj} \mu_j}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j.$$

Hence the variations δa_{sj} are to be connected by the condition

$$(III, 12) \quad 0 = \kappa \sum_{s=1}^n \sum_{j=1}^{\nu} \delta a_{sj} \left\{ \frac{\mu_j}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j + \mu_j \sum_{i=s+1}^{n-1} \frac{\sum_{j=1}^{\nu} a_{ij} \mu_j}{r_i} \right\}$$

that is to be added to the equations (III, 2) and (III, 3) which run as follows when i is substituted by s

$$(III, 2) \quad 0 = \sum_{j=1}^{\nu} \sum_{s=1}^n \delta a_{sj} \left\{ \log \frac{n a_{sj}}{N_j} + 1 + \frac{1}{2 a_{sj}} \right\}$$

$$(III, 3) \quad 0 = \sum_{s=1}^n \delta a_{sj}$$

To solve the a_{sj} we again use the method of indetermined multipliers. On multiplying the equation (III, 12) by m , the ν equations (III, 3) by different factors l_j and adding the $\nu+1$ equations to equation (III, 2) we obtain

$$0 = \sum_{s=1}^n \sum_{j=1}^{\nu} \delta a_{sj} \left\{ \log \frac{n a_{sj}}{N_j} + 1 + \frac{1}{2 a_{sj}} + l_j + m \kappa \left(\frac{1}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j + \sum_{i=s+1}^{n-1} \frac{\sum_{j=1}^{\nu} a_{ij} \mu_j}{r_i} \right) \right\}$$

and the quantities a_{sj} are to satisfy the equations

$$0 = \log \frac{n a_{sj}}{N_j} + 1 + \frac{1}{2 a_{sj}} + l_j + m \kappa \mu_j \left(\frac{1}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j + \sum_{i=s+1}^{n-1} \frac{\sum_{j=1}^{\nu} a_{ij} \mu_j}{r_i} \right).$$

To these s, j equations we join the j equations (III, 1) and equation (III, 11) and thus have $s, j + j + 1$ equations to determine the $s, j + j + 1$ unknown quantities a_{sj} , l_j and m . But the quantities l_j and m can just as well be accepted as characterising the distribution as the N_j 's and h . Having observed that the N_j 's and l_j 's are not independent of one another I will, however, retain the N_j 's, l_j 's and m in our equations. A simple consideration of some properties of the distribution satisfying our equation will enable us later on to introduce a new set of quantities more suitably characterising the distribution.

As the a_{sj} 's are so great that we are able to neglect $\frac{1}{2 a_{sj}}$ when compared with unity, we obtain the equation

$$(III, 13) \quad 0 = \log \frac{n a_{sj}}{N_j} + 1 + l_j + m \kappa \mu_j \left(\frac{1}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j + \sum_{i=s+1}^{n-1} \frac{\sum_{j=1}^{\nu} a_{ij} \mu_j}{r_i} \right).$$

In order to find this equation we searched for a distribution possessing the extremum of probability among all distributions having constant Ω and a constant number of individuals. The s, j equations condensed in the relation (III, 13) can be expressed symbolically by

$$0 = \delta \sum_{j=1}^{\nu} K_{Aj} + \delta \sum_{j=1}^{\nu} l_j \sum_{s=1}^n a_{sj} + m \delta \Omega.$$

It is of interest to find these equations by a somewhat different deduction.

If the state of a system is determined by equations of the form

$$\frac{d^2 x}{dt^2} = X$$

$$\frac{d^2 y}{dt^2} = Y$$

$$\frac{d^2 z}{dt^2} = Z$$

where x, y, z are positional coordinates of particles in the system, and X, Y, Z are the acting forces, and if these forces are only dependent on the positional coordinates, the system is said to be conservative. Then the double sum taken over all particles and directions of the axis

$$\Sigma \Sigma (X dx + Y dy + Z dz)$$

is a total differential, dU , and the function U is called the potential energy of the system (eventually increased by a constant). According to the theorem of LEJEUNE-DIRICHLET, the system is in a state of equilibrium if δU is vanishing, and, furthermore, it is possible to distinguish if the equilibrium is stable or not by investigating whether the extremum of U is maximum or minimum.

Now let our system of stars be determined in relation to a rectangular system of coordinates x, y, z and let the corresponding velocities be designed u, v, w . Let

$$v_j f_j (x, y, z, u, v, w, t) dx dy dz du dv dw, \text{ where } \iiint f_j du dv dw = 1$$

design the number of stars of mass μ_j whose coordinates and velocities at the time t satisfy the conditions

$$x - \frac{1}{2} dx < x < x + \frac{1}{2} dx$$

$$y - \frac{1}{2} dy < y < y + \frac{1}{2} dy$$

$$z - \frac{1}{2} dz < z < z + \frac{1}{2} dz$$

$$u - \frac{1}{2} du < u < u + \frac{1}{2} du$$

$$v - \frac{1}{2} dv < v < v + \frac{1}{2} dv$$

$$w - \frac{1}{2} dw < w < w + \frac{1}{2} dw$$

Out of these stars there is, during the time interval dt , a net loss of a number

$$\frac{\partial}{\partial x} [v_j f_j (x, y, z, u, v, w, t) dx dy dz du dv dw u dt]$$

taking place through the faces perpendicular to the axis of x limiting the element of volume. By taking the sums over all three pairs of faces and over all values of velocities the total loss of molecules is found to be

$$dx dy dz dt \iiint \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) v_j f_j du dv dw.$$

If

$$u_0 = \iiint u f_j du dv dw \quad \text{etc.}$$

the loss is expressed by

$$\left(\frac{\partial}{\partial x} (v_j u_0) + \frac{\partial}{\partial y} (v_j v_0) + \frac{\partial}{\partial z} (v_j w_0) \right) dx dy dz dt$$

As this loss is also expressed by

$$- \frac{\partial v_j}{\partial t} dx dy dz dt$$

we get an equation of continuity

$$\frac{\partial v_j}{\partial t} + \frac{\partial}{\partial x} (v_j u_0) + \frac{\partial}{\partial y} (v_j v_0) + \frac{\partial}{\partial z} (v_j w_0) = 0.$$

When the

$$dx dy dz dt \iiint \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z} \right) v_j f_j du dv dw$$

stars leave the element of volume it also suffers a loss of momentum parallel to the axis of x and amounting to

$$\mu_j dx dy dz dt \iiint \left(u^2 \frac{\partial}{\partial x} + u v \frac{\partial}{\partial y} + u w \frac{\partial}{\partial z} \right) v_j f_j du dv dw.$$

If

$$\overline{u^2} = \iiint u^2 f_j du dv dw \quad \text{etc.}$$

the loss of momentum is expressed by

$$\mu_j dx dy dz dt \left(\frac{\partial}{\partial x} (v_j \overline{u^2}) + \frac{\partial}{\partial y} (v_j \overline{uv}) + \frac{\partial}{\partial z} (v_j \overline{uw}) \right).$$

Let X be the force acting on a certain star in the direction of the axis of x . During the time dt the momentum of the star is then increased by $X dt$. Taking

the sum of the X 's over all stars we find the total gain of momentum parallel to the axis of x

$$dt \left(\Sigma X - \mu_j dx dy dz \frac{\partial}{\partial x} (v_j \overline{u^2}) + \frac{\partial}{\partial y} (v_j \overline{uv}) + \frac{\partial}{\partial z} (v_j \overline{uw}) \right)$$

This gain is also expressed by

$$\frac{\partial}{\partial t} (v_j u_0) \mu_j dx dy dz dt$$

and we obtain the equation

$$\left(\frac{\partial}{\partial t} (v_j u_0) + \frac{\partial}{\partial x} (v_j \overline{u^2}) + \frac{\partial}{\partial y} (v_j \overline{uv}) + \frac{\partial}{\partial z} (v_j \overline{uw}) \right) \mu_j dx dy dz = \Sigma X.$$

Now introduce

$$\begin{aligned} u &= u_0 + U \\ v &= v_0 + V \\ w &= w_0 + W. \end{aligned}$$

Thus

$$\begin{aligned} \overline{u^2} &= u_0^2 + \overline{U^2} \\ \overline{uv} &= u_0 v_0 + \overline{UV} \\ \overline{uw} &= u_0 w_0 + \overline{UW}. \end{aligned}$$

As the equation of continuity provides us with

$$\begin{aligned} \frac{\partial}{\partial t} (v_j u_0) &= v_j \frac{\partial}{\partial t} u_0 + u_0 \frac{\partial v_j}{\partial t} \\ &= v_j \frac{\partial}{\partial t} u_0 - u_0 \left(\frac{\partial}{\partial x} (v_j u_0) + \frac{\partial}{\partial y} (v_j v_0) + \frac{\partial}{\partial z} (v_j w_0) \right) \end{aligned}$$

we find

$$\begin{aligned} v_j \left(\frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x} + v_0 \frac{\partial}{\partial y} + w_0 \frac{\partial}{\partial z} \right) u_0 \mu_j dx dy dz &= \Sigma X - \left(\frac{\partial}{\partial x} (v_j \overline{U^2}) + \right. \\ &\quad \left. + \frac{\partial}{\partial y} (v_j \overline{UV}) + \frac{\partial}{\partial z} (v_j \overline{UW}) \right) \mu_j dx dy dz, \end{aligned}$$

Now, if the irregular forces are neglected the stars in the element of volume will still be influenced by gravitational forces which are then proportional to the mass

of every star. Let Ξ stand for the gravitational force per unit mass. Then we have

$$\Sigma X = \Xi \mu_j v_j dx dy dz$$

and the equation of momentum becomes

$$\frac{d u_0}{dt} = \Xi - \frac{1}{v_j} \left(\frac{\partial}{\partial x} (v_j \overline{U^2}) + \frac{\partial}{\partial y} (v_j \overline{UV}) + \frac{\partial}{\partial z} (v_j \overline{UW}) \right).$$

According to the preceding chapter we are now able to put

$$\begin{aligned} \mu_j \overline{U^2} &= a \text{ constant} = \eta \\ \overline{UV} &= \overline{UW} = 0. \end{aligned}$$

Thus we have the equation

$$\frac{d u_0}{dt} = \Xi - \frac{\eta}{\mu_j v_j} \frac{\partial}{\partial x} v_j.$$

This equation is comparable to the usual hydrodynamical equation

$$\frac{d u_0}{dt} = \Xi - \frac{1}{v_j} \frac{\partial}{\partial x} p$$

where p is the pressure.

On discussing the viriel of CLAUSIUS MAXWELL¹⁾ had already remarked that the pressure in a mass of gas could not be produced entirely by the *molecular forces* but depended partly or wholly on the *motion* of the molecules. It is, therefore, not at all surprising that we found the motion of the stars, when the irregular forces are neglected, to be still analogous to the motion in a mass of gas.

Now consider again the spherically distributed system of stars that we have treated on the preceding pages. We find that the motion within it is governed by the equation

$$\frac{d u_0}{dt} = \Xi - \frac{\eta}{\mu_j v_j} \frac{\partial}{\partial x} v_j$$

and by similar equations for the other two directions of axis. Putting $\varphi(x, y, z)$ for the potential of the gravitational forces we have

$$\frac{d u_0}{dt} = \frac{\partial}{\partial x} \left(\varphi - \frac{\eta}{\mu_j} \log v_j \right).$$

Similar equations are valid for all other groups of masses.

¹⁾ J. C. MAXWELL: On the Dynamical Evidence of the Molecular Constitution of Bodies. The Sc. Papers of J. C. MAXWELL, edited by W. D. NIVEN (Cambridge 1890), Vol. II, p. 422.

As the system is spherically symmetrical these equations can be substituted by those giving the motion along the radius

$$\frac{d^2 r}{dt^2} = \frac{\partial}{\partial r} \left(\varphi - \frac{\eta}{\mu_j} \log v_j \right)$$

where $\frac{dr}{dt}$ stands for the stream-velocity, that is the mean velocity, along the radius. Apply this equation to a displacement of the a_{sj} stars of the cell s having mass μ_j and we find the work by this displacement to be

$$a_{sj} \mu_j \left(-\varphi + \frac{\eta}{\mu_j} \log a_{sj} + \text{a constant } A_{sj} \right)$$

where A_{sj} is the potential energy previous to the displacement. But as we have

$$\varphi = \frac{\kappa}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j$$

the exhaustion of this potential energy is

$$\kappa \frac{a_{sj} \mu_j}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j - \eta a_{sj} \log a_{sj} - a_{sj} \mu_j A_{sj}.$$

On taking the sum over all s and j we get the total exhaustion of potential energy of the stream-motion within the system

$$\Omega = \sum_{s=1}^n \sum_{j=1}^{\nu} \eta a_{sj} \log a_{sj} - \sum_{s=1}^n \sum_{j=1}^{\nu} a_{sj} \mu_j A_{sj}.$$

The distribution of equilibrium is then obtained by combining the conditions

$$0 = \delta \left(\Omega - \sum_{s=1}^n \sum_{j=1}^{\nu} \eta a_{sj} \log a_{sj} - \sum_{s=1}^n \sum_{j=1}^{\nu} a_{sj} \mu_j A_{sj} \right)$$

and

$$\delta \sum_{s=1}^n a_{sj} = N_j \quad (j = 1, \dots, \nu).$$

When these variations are expressed by δa_{sj} and the variations of the last equations multiplied by l_j 's we obtain by addition

$$0 = \sum_{s=1}^n \sum_{j=1}^{\nu} \delta a_{sj} \left[\kappa \left\{ \frac{\mu_j}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j + \mu_j \sum_{i=s+1}^{n-1} \frac{\sum_{j=1}^{\nu} a_{ij} \mu_j}{r_i} \right\} - \eta (\log a_{sj} + 1) - \mu_j A_{sj} + l_j \right].$$

The state of equilibrium is thus determined by the equations

$$0 = \eta (\log a_{sj} + 1) + \mu_j A_{sj} - l_j - \kappa \left\{ \frac{\mu_j}{r_s} \sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j + \mu_j \sum_{i=s+1}^{n-1} \frac{\sum_{j=1}^{\nu} a_{ij} \mu_j}{r_i} \right\}.$$

On choosing convenient relations between the constants in these equations and those entering into the equation (III, 13) the two systems are seen to be identical. The search for the state of equilibrium has brought forth the same condition as the search for the most probable system (both are based on the assumption that equipartition of energy exists).

As the distribution is treated statistically we must here observe that the condition giving the mean values of the distribution when in a state of equilibrium, does not guarantee the system to be really in equilibrium. If the equilibrium is unstable the slightest deviation from the state of equilibrium will destroy the distribution. In order to accept a statistical state of equilibrium as being physically in equilibrium we must also require the equilibrium to be stable. It seems possible by closer considerations of the extremum, determined by the preceding equation, to determine whether it is a maximum or a minimum and thus get an answer to the question regarding the stability of the system. For the moment, however, I will refrain from taking up this question for discussion.

Now let us return to equation (III, 13). R designs the radius of the sphere enclosing all the stars considered. Let W be the volume of this sphere. Then the volume of every cell is

$$\frac{W}{n}$$

and

$$\frac{n a_{sj}}{W}$$

gives the density of stars having mass μ_j at the distance r_s from the centre. Let us introduce

$$f_j(r_s) = \frac{n a_{sj}}{W}$$

which gives

$$a_{sj} = \frac{W}{n} f_j(r_s).$$

If $r_{s+\frac{1}{2}}$ is the radius of the spherical surface separating the cell s from the cell $s+1$ we have

$$\frac{W}{n} = \int_{r_s - \frac{1}{2}}^{r_s + \frac{1}{2}} 4 \pi r^2 dr$$

and

$$a_{sj} = f_j(r_j) \int_{r_{s-\frac{1}{2}}}^{r_{s+\frac{1}{2}}} 4\pi r^2 dr.$$

On considering the quantities $f_j(r_s)$ as being continuously variable functions of r and further treating the cells as thin shells, we may put

$$a_{sj} = \int_{r_{s-\frac{1}{2}}}^{r_{s+\frac{1}{2}}} 4\pi r^2 f_j(r) dr.$$

Following the same assumption we will transform the expressions of the sums of the equation (III, 13) and thus obtain

$$\sum_{i=1}^{s-1} \sum_{j=1}^{\nu} a_{ij} \mu_j = \sum_{i=1}^{s-1} \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} 4\pi r^2 \sum_{j=1}^{\nu} \mu_j f_j(r) dr = \int_{r_{\frac{1}{2}}}^{r_{s-\frac{1}{2}}} 4\pi r \sum_{j=1}^{\nu} \mu_j f_j(r) dr$$

and

$$\sum_{i=s+1}^{n-1} \frac{\sum_{j=1}^{\nu} a_{ij} \mu_j}{r_i} = \sum_{i=s+1}^{n-1} \int_{r_{i-\frac{1}{2}}}^{r_{i+\frac{1}{2}}} 4\pi r \sum_{j=1}^{\nu} \mu_j f_j(r) dr = \int_{r_{s+\frac{1}{2}}}^{r_{n-\frac{1}{2}}} 4\pi r \sum_{j=1}^{\nu} \mu_j f_j(r) dr.$$

Supposing the cells to be thin shells (though the a_{sj} 's are great numbers), we neglect the integrals between the limits $r_{\frac{1}{2}} = 0, r_{s+\frac{1}{2}} = r_{s-\frac{1}{2}}$ and $R = r_{n-\frac{1}{2}}$ and finally drop the index s that has lost its significance. We thus obtain

$$(III, 14) \quad 0 = \log \frac{W}{N_j} f_j(r) + 1 + l_j + m \kappa \mu_j \left(\frac{1}{r} \int_0^r 4\pi r^2 \sum_{j=1}^{\nu} \mu_j f_j(r) dr + \right. \\ \left. + \int_0^R 4\pi r \sum_{j=1}^{\nu} \mu_j f_j(r) dr \right) \quad (j = 1, \dots, \nu).$$

But

$$\frac{1}{r} \int_0^r 4\pi r^2 \sum_{j=1}^{\nu} \mu_j f_j(r) dr + \int_r^R 4\pi r \sum_{j=1}^{\nu} \mu_j f_j(r) dr$$

is the potential $\varphi(r)$, and solving $\varphi(r)$ from the equation (III, 14) we have

$$(III, 15) \quad \varphi(r) = - \frac{\log \frac{W}{N_j} f_j(r) + 1 + l_j}{m \kappa \mu_j} = - \frac{1}{m \kappa \mu_j} \log \left(\frac{W}{N_j} e^{1+l_j} f_j(r) \right).$$

The potential of gravitation must, however, satisfy the equation of Poisson

$$\nabla^2 \varphi(r) = -4\pi \varrho$$

where ϱ is the density. As we have assumed the distribution of the masses to be spherically symmetrical we have

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr}.$$

Applying the Laplacian operator to the expressions (III, 15) for $\varphi(r)$ and introducing

$$\varrho = \sum_{j=1}^{\nu} \mu_j f_j(r)$$

we find the equations

$$(III, 16) \quad \left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right) \left(- \frac{1}{m \kappa \mu_j} \log \left(\frac{W}{N_j} e^{1+l_j} f_j(r) \right) \right) = -4\pi \sum_{j=1}^{\nu} \mu_j f_j(r) \quad (j = 1, \dots, \nu)$$

which determine the laws of distribution of the masses of the ν different classes of sizes.

As

$$\frac{d}{dr} \log \frac{W}{N_j} e^{1+l_j} = 0$$

the equations (III, 16) give

$$(III, 17) \quad \frac{d^2 \log f_j(r)}{dr^2} + \frac{2}{r} \frac{d \log f_j(r)}{dr} = 4\pi m \kappa \mu_j \sum_{j=1}^{\nu} \mu_j f_j(r) \quad (j = 1, \dots, \nu).$$

Each one of these ν equations contains in its left membrum one of the unknown functions but in its right membrum all the unknown functions. By considering the equations (III, 15) we easily get relations expresseing $\nu - 1$ of the functions as functions of the remaining one. In the ν equations (III, 15) the left members are identical and consequently the right members must also be alike. We thus obtain

$$\frac{1}{m \kappa \mu_i} \left(\log \frac{W}{N_i} e^{1+l_i} f_i(r) \right) = \frac{1}{m \kappa \mu_j} \left(\log \frac{W}{N_j} e^{1+l_j} f_j(r) \right) \quad (i = 1, \dots, \nu).$$

Hence

$$(III, 18) \quad \left(\frac{W}{N_i} e^{1+l_i} f_i(r) \right)^{1/\mu_i} = \left(\frac{W}{N_j} e^{1+l_j} f_j(r) \right)^{1/\mu_j}$$

These relations were used by v. ZEIPPEL and LINDGREN (1921).

If we introduce

$$(III, 19) \quad C_{ij} = \frac{N_i}{W} \left(\frac{W}{N_j} \right)^{\frac{\mu_i}{\mu_j}} e^{-(1+l_i) + \frac{\mu_i}{\mu_j} (1+l_j)}$$

we have

$$(III, 20) \quad f_i(r) = C_{ij} [f_j(r)]^{\frac{\mu_i}{\mu_j}}$$

Then

$$\sum_{j=1}^{\nu} \mu_j f_j(r) = \sum_{i=1}^{\nu} \mu_i C_{ij} [f_j(r)]^{\frac{\mu_i}{\mu_j}}$$

and the equations (III, 17) become

$$(III, 21) \quad \frac{d^2 \log f_j(r)}{dr^2} + \frac{2}{r} \frac{d \log f_j(r)}{dr} = 4 \pi m \kappa \mu_j \sum_{i=1}^{\nu} \mu_i C_{ij} [f_j(r)]^{\frac{\mu_i}{\mu_j}},$$

where both the right and the left membrum only contain one of the unknown functions. In order to find all functions of distribution we need only solve one of the ν equations (III, 21). The other $\nu - 1$ functions are then found by aid of the relations (III, 18).

The differential equation of distribution of equal masses is easily found from (III, 21). Suppose

$$\mu_j = 0 \quad (j = 2, 3, \dots, \nu)$$

and

$$\mu_1 = 1$$

then we immediately have

$$(III, 22) \quad \frac{d^2 \log f_1(r)}{dr^2} + \frac{2}{r} \frac{d \log f_1(r)}{dr} = 4 \pi m \kappa f_1(r).$$

This is the well-known equation of isothermal distribution that has been thoroughly studied by EMDEN (*Gaskugeln*, p. 131. equat. II). The equation (III, 22) cannot be solved in finite terms and it seems just as impossible to find a solution in finite terms for the equation (III, 21).

The parameters C_{ij} entering by relation (III, 19) the equation (III, 21) are functions of the volume W . It is, however, possible to substitute for these para-

meters new ones that are independent of W . Let W' be the volume of another sphere within W having the same centre. At a point P , whose distance from the centre is r , $\varphi(r)$ and $f_j(r)$ have certain values determined by the equations (III, 21) and (III, 15). The distribution within W' might, however, be determined by parameters having some relations to W' . As the distribution is still the same we must find the same values for $\varphi(r)$ and $f_j(r)$. As P is situated within W we get from the equation (III, 15)

$$\varphi(r) = - \frac{1}{m \kappa \mu_j} \log \left(\frac{W}{N_j} e^{I + l_j} f_j(r) \right)$$

and as P belongs to the volume W' we find in the same manner

$$\varphi(r) = - \frac{1}{m' \kappa \mu_j} \log \left(\frac{W'}{N_j'} e^{I + l_j'} f_j(r) \right)$$

where the apostrophes indicate quantities related to the volume W' .

From these two equations we obtain

$$\left(\frac{W}{N_j} e^{I + l_j} \right)^{\frac{1}{m}} \left(f_j(r) \right)^{\frac{1}{m}} = \left(\frac{W'}{N_j'} e^{I + l_j'} \right)^{\frac{1}{m'}} \left(f_j(r) \right)^{\frac{1}{m'}}$$

or

$$\frac{\left(\frac{W}{N_j} e^{I + l_j} \right)^{\frac{1}{m}}}{\left(\frac{W'}{N_j'} e^{I + l_j'} \right)^{\frac{1}{m'}}} = \left(f_j(r) \right)^{\frac{1}{m'} - \frac{1}{m}}.$$

Similarly, in respect to another point Q within both volumes we find

$$\frac{\left(\frac{W}{N_j} e^{I + l_j} \right)^{\frac{1}{m}}}{\left(\frac{W'}{N_j'} e^{I + l_j'} \right)^{\frac{1}{m'}}} = \left(f_j(r') \right)^{\frac{1}{m'} - \frac{1}{m}}$$

where r' designs the distance between Q and the centre. In the last two preceding equations the left members are identical as all quantities involved are constants of the distributions. In the right members $f_j(r)$ and $f_j(r')$ are, however, unequal. Thus we must have

$$m = m'.$$

This result confirms a result of the analysis given on page 34 *viz.* that m is independent of W as it is the constant of the equipartition of energy.

We further find

$$(III, 23) \quad \frac{W}{N_j} e^{I + l_j} = \frac{W'}{N_j'} e^{I + l_j'} = \frac{1}{K_j}$$

where a new parameter K_j independent of W is introduced.

The equation (III, 19) is then transformed into

$$C_{ij} = K_i \left(\frac{1}{K_j} \right)^{\frac{\mu_i}{\mu_j}}$$

and equation (III, 20) into

$$(III, 24) \quad f_i(r) = K_i \left(\frac{1}{K_j} \right)^{\frac{\mu_i}{\mu_j}} \left(f_j(r) \right)^{\frac{\mu_i}{\mu_j}}.$$

Finally, the equation (III, 21) is transformed into

$$(III, 25) \quad \frac{d^2 \log f_j(r)}{dr^2} + \frac{2}{r} \frac{d \log f_j(r)}{dr} = 4 \pi m \kappa \mu_j \sum_{i=1}^{\nu} \mu_i K_i \left(\frac{1}{K_j} f_j(r) \right)^{\frac{\mu_i}{\mu_j}},$$

The equations (III, 24) and (III, 25) determine the most probable distribution by means of parameters that are independent of the volume W . We are permitted to extend the volume W as far as the distribution reaches in space and the distribution is still given by our equations.

Let us find some properties of the functions $f_j(r)$ when the volume occupied by stars distributed in the most probable manner extends beyond every limit.

Suppose that μ_1 is the smallest of the μ_i 's that is

$$\mu_1 < \mu_s \quad (s = 2, \dots, \nu)$$

and let us put $j = 1$ in the equation (III, 25). It is a well-known fact (vide Ritter. loc. cit.) that the solution of the equation of distribution of equal masses

$$(III, 26) \quad \frac{d^2 \log f_1(r)}{dr^2} + \frac{2}{r} \frac{d \log f_1(r)}{dr} = 4 \pi m \kappa \mu_1^2 f_1(r)$$

is vanishing like $\frac{1}{r^2}$ for great values of r . But as $\frac{\mu_j}{\mu_1}$ is always greater than unity the equation (III, 25) becomes (III, 26) when r goes to infinity. Hence we conclude that the solution of (III, 25) also vanishes as $\frac{1}{r^2}$ does for great values of r .

The number of stars N_1 situated within the sphere having radius R and having the mass μ_1 is given by the expression

$$N_1 = \int_0^R 4 \pi r^2 f_1(r) dr$$

Hence we have

$$\lim_{R \rightarrow \infty} L N_1 = \int_0^{\infty} 4 \pi r^2 f_1(r) dr = \infty$$

because $4 \pi r^2 f_1(r)$ approaches a constant value when r goes to infinity.

In the same manner we find the number N_i of stars of mass μ_i

$$\lim_{R \rightarrow \infty} L N_i = \int_0^{\infty} 4 \pi r^2 K_i \left(\frac{1}{K_1} f_1(r) \right) \frac{\mu_i}{\mu_1} dr.$$

If $\frac{\mu_i}{\mu_j} > \frac{3}{2}$ the expression

$$4 \pi r^2 K_i \left(\frac{1}{K_1} f_1(r) \right) \frac{\mu_i}{\mu_1}$$

approaches zero just as $\frac{1}{r^{1+a}}$ does, where $a > 0$, when r goes to infinity. Consequently the integral has a finite value.

As we have previously supposed the quantities a_{sj} to be great numbers, the extension of R to infinity is not strictly valid when $f_1(r)$ approaches zero. I think, however, that the following conclusions may be drawn from our analysis.

The system of stars possessing the most probable distribution does not present any free surface, but extends to infinity. If the distribution of the masses of the stars reaches over a greater ratio than 3:2 the smallest stars must necessarily be innumerable but the number of stars belonging to the classes of great masses can be finite.

Now suppose that a globular cluster is built up by stars belonging to classes of masses that reach over a greater ratio than 3:2. Further suppose that the luminosities of the smallest stars are too weak to enable these stars to be seen from the earth. Finally, suppose that the distribution of the stars in the cluster is the most probable one. Then the law of distribution, as observed from the earth, can present the properties of extending to infinity and of possessing a finite number of individuals. As has been proved by EMDEN and emphasized by EDDINGTON, only one of the adiabatic distributions presents these properties, i. e. the one having $n=5$ ($\gamma=1.2$). If the distribution in the globular cluster is assumed to be adiabatic, the result found, namely, that $n=5$, seems to be a reasonable one, and is explained by the fact that the distribution is in reality the most probable one.

As far as I know there does not as yet exist enough observations on globular clusters to permit a closer scrutiny of the hypothetical explanation regarding the observed value $n=5$.

I will, however, apply the equations (III, 25) to an open cluster, namely, the one studied by v. ZEIPEL and LINDGREN, M 37.

CHAPTER IV.

Application of Theory to the Open Cluster M 37.

As has been mentioned in the Introduction, v. ZEIPEL and J. LINDGREN have investigated the distribution of stars in the cluster M 37 (N. G. C. 2099), (A. R. = $5^h 46^m$, D = $+32^\circ 32'$). Their catalogue contains the positions and photographic magnitudes of 2110 stars within the cluster. Color-indices are also given for 1885 stars. The stars are divided according to their magnitudes and color-indices into groups p , q , u . Besides these stars others of higher magnitudes ($15.7^M-16.5^M$) are counted and form a fourth group w . The p -stars are interpreted as g -giants, the q -stars as b - and a -stars, the u -stars as f -dwarfs and the w -stars as g -dwarfs.

The distributions in space $f(r)$ of the four types of stars belonging to the central parts of the cluster are given in the appended table 1, taken from table 40 in the work quoted.

TABLE 1.

$$\log f(r) + 10.000$$

r	$\log r^2$	p	q	u	w
0	$-\infty$	9.233	9.604	8.901	
1	0.000	9.182	9.581	8.882	
2	0.602	8.972	9.505	8.802	8.313
3	0.954	8.556	9.367	8.693	8.264
5	1.398	7.927	8.993	8.476	8.154
8	1.806	7.350	8.526	8.161	7.960

r is measured in millimeters on the photographic plate. (8 mm. = $6'.2$).

We adopt for our purpose the symbols used by v. ZEIPEL and LINDGREN and explained in the following schedule:

Group	Mass	Density in Space
p	μ_1	$f_1(r)$
q	1	$f(r)$
u	μ_2	$f_2(r)$
w	μ_3	$f_3(r)$

It is here assumed that a star's mass is a function of its absolute magnitude and color-index and thus a certain mass-value is attributed to every group. (As all stars in the cluster are at approximately the same distance, the absolute magnitude is reduced to the apparent magnitude by adding a constant).

The Maxwellian distribution, that has been proved to be the most probable one for a gaseous mass, is assumed by V. ZEPEL and J. LINDGREN to hold true also for a stellar system where the collisions (and passages) are rare enough to be neglected. As sketched out in the Introduction, the following relations exist

$$(IV, 1) \quad f_k(r) = C_k [f(r)]^{\mu_k} \quad (k = 1, 2, 3)$$

(compare equations (III, 20) and (III, 24)) or

$$-\log f_k(r) = c_k + \mu_k [-\log f(r)]$$

where

$$c_k = -\log C_k$$

is independent of r . Applying these relations to the observed numbers in table 1. the authors obtain a system of equations which can be solved by the method of least squares and gives the following values of c_k and μ_k (the value $\log f_1(8) = 7.350 - 10$ is excluded in order to get a better fit)

$$\begin{array}{llll} c_1 = -0.043, & c = 0.000, & c_2 = 0.851, & c_3 = 1.501 \\ \mu_1 = 2.152, & \mu = 1.000, & \mu_2 = 0.673, & \mu_3 = 0.360. \end{array}$$

The arbitrary unit of mass is chosen in such a manner that we obtain $\mu = 1$.

The good agreement between the mass values and those obtained by other methods for the stars of the same spectral types, and the smallness of the mean errors in the determination of the μ_k 's (being ± 0.12 for μ_1 , ± 0.020 for μ_2 and ± 0.012 for μ_3) are *a posteriori* proofs bearing out, according to the authors, the hypothesis of the Maxwellian distribution. I think, however, that it is of interest to consider whether the functions $f_k(r)$ (and not only their relations) fulfil the conditions set up for the most probable distribution, and thus, satisfy the equation (III, 25).

When the equation (III, 25) is written with the symbols used in this chapter and applied to the stars of group q we have

$$(IV, 2) \quad \frac{d^2 \log f}{dr^2} + \frac{2}{r} \frac{d \log f}{dr} = 4 \pi m \times \left(f + \mu_1 C_1 f^{\mu_1} + \mu_2 C_2 f^{\mu_2} + \mu_3 C_3 f^{\mu_3} \right)$$

This equation contains the natural logarithms and the transition into decadic logarithms is to be effected by multiplying the right member by a numerical factor. As we already have an arbitrary constant, m , in the right member we take the numerical factor together with the parameter m and introduce a new parameter m' . The *logs* then design decadic logarithms.

Further introducing the numerical values of μ_k and C_k we find

$$(IV, 3) \quad \frac{d^2 \log f}{dr^2} + \frac{2}{r} \frac{d \log f}{dr} = 4 \pi m' \kappa \left(f + 2.375 f^{2.152} + 0.095 f^{0.673} + 0.011 f^{0.360} \right).$$

In order to find a solution of this equation we are obliged to use some method of numerical integration, and I have chosen the one exposed by Prof. CHARLIER in *Mechanik des Himmels* (Section 8). The equation, however, contains an unknown parameter, m' , that cannot be determined until the theoretical distribution is compared with the observed one. Introduce, however, a new independent variable ϱ connected to r by the relation

$$(IV, 4) \quad \varrho = r \sqrt{-4 \pi m' \kappa}$$

(it will appear later on that m' is really a negative quantity). Then equation (IV, 3) becomes

$$(IV, 5) \quad \frac{d^2 \log f}{d\varrho^2} + \frac{2}{\varrho} \frac{d \log f}{d\varrho} = - \left(f + 2.375 f^{2.152} + 0.095 f^{0.673} + 0.011 f^{0.360} \right)$$

which is equivalent to the system

$$(IV, 6) \quad \begin{aligned} \frac{d \log f}{d\varrho} &= y \\ \frac{dy}{d\varrho} &= - \left(\frac{2}{\varrho} y + f + 2.375 f^{2.152} + 0.095 f^{0.673} + 0.011 f^{0.360} \right). \end{aligned}$$

We are obliged to choose a system of initial values of ϱ , y and $\log f$. We will start from the point $\varrho = 0$. Indeed, this is a singular point of our equation, but as we do not need the complete solution we will limit ourselves to the regular one. Observation gives $\log f(0) = 9.604 - 10$ while the solution by the method of least squares, calculated by v. ZEIPPEL and LINDGREN, changes this value to $9.613 - 10$. We hence start the numerical calculation with the values

$$\varrho=0, \quad y=0, \quad \log f(0) = 9.613-10.$$

The calculation is based on the process recommended by CHARLIER. Using the formulas (3) and (9) (loc. cit. p. 51 and p. 49) each value is evaluated twice from two series of differences. The value of ω (the difference between the ϱ 's for the two consecutively calculated values of $\log f$) was chosen to be 0.02 when ϱ was small, and greater when ϱ grew greater. The result is given in table 2. This table also gives the values of $\log f_1$, $\log f_2$ and $\log f_3$ obtained from $\log f$ by means of the equations (IV, 1).

TABLE 2.

ϱ	$\log \varrho^2$	$\log f_1$	$\log f$	$\log f_2$	$\log f_3$
0.00	— ∞	9.210—10	9.613—10	8.888—10	8.360—10
0.16	8.408—10	9.202	9.609	8.886	8.358
0.32	9.010	9.176	9.597	8.878 ^c	8.354
0.48	9.362	9.146	9.583	8.868	8.349
0.64	9.612	9.096	9.560	8.853	8.341
0.80	9.806	9.038	9.533	8.835	8.331
1.12	0.098	8.894	9.466	8.790	8.307
1.44	0.317	8.726	9.388	8.737	8.279
1.76	0.491	8.550	9.306	8.682	8.249
2.08	0.636	8.367	9.221	8.625	8.219
2.40	0.760	8.184	9.136	8.568	8.188
2.72	0.869	8.007	9.054	8.512	8.158
3.04	0.966	7.835	8.974	8.459	8.130
3.68	1.132	7.510	8.823	8.357	8.075
4.32	1.271	7.211	8.684	8.263	8.025
5.28	1.445	6.809	8.497	8.137	7.958

The factor connecting ϱ to r remains to be determined. As

$$\varrho = r \sqrt{-4 \pi m' \kappa}$$

we have

$$(IV, 7) \quad \log \varrho^2 = \log r^2 + \log (-4 \pi m' \kappa)$$

If we draw a diagram giving the observed values of $\log f$ as functions of $\log r^2$ and another giving the calculated values of $\log f$ as functions of $\log \varrho^2$ we see from equation (IV, 7) that the two diagrams can be made to correspond by simply adjusting the origin of the one diagram along the $\log r^2$ -axis of the other. When this operation is carried out in respect to the stars of group q a preliminary value is found

$$\log (-4 \pi m' \kappa) = -0.5.$$

Let the exact value be $-0.5 - \varepsilon$. We can then determine ε from the equations of the form

$$\log f_{obs}(\log r^2) = \log f_{cal}(\log \varrho^2 + 0.5 + \varepsilon)$$

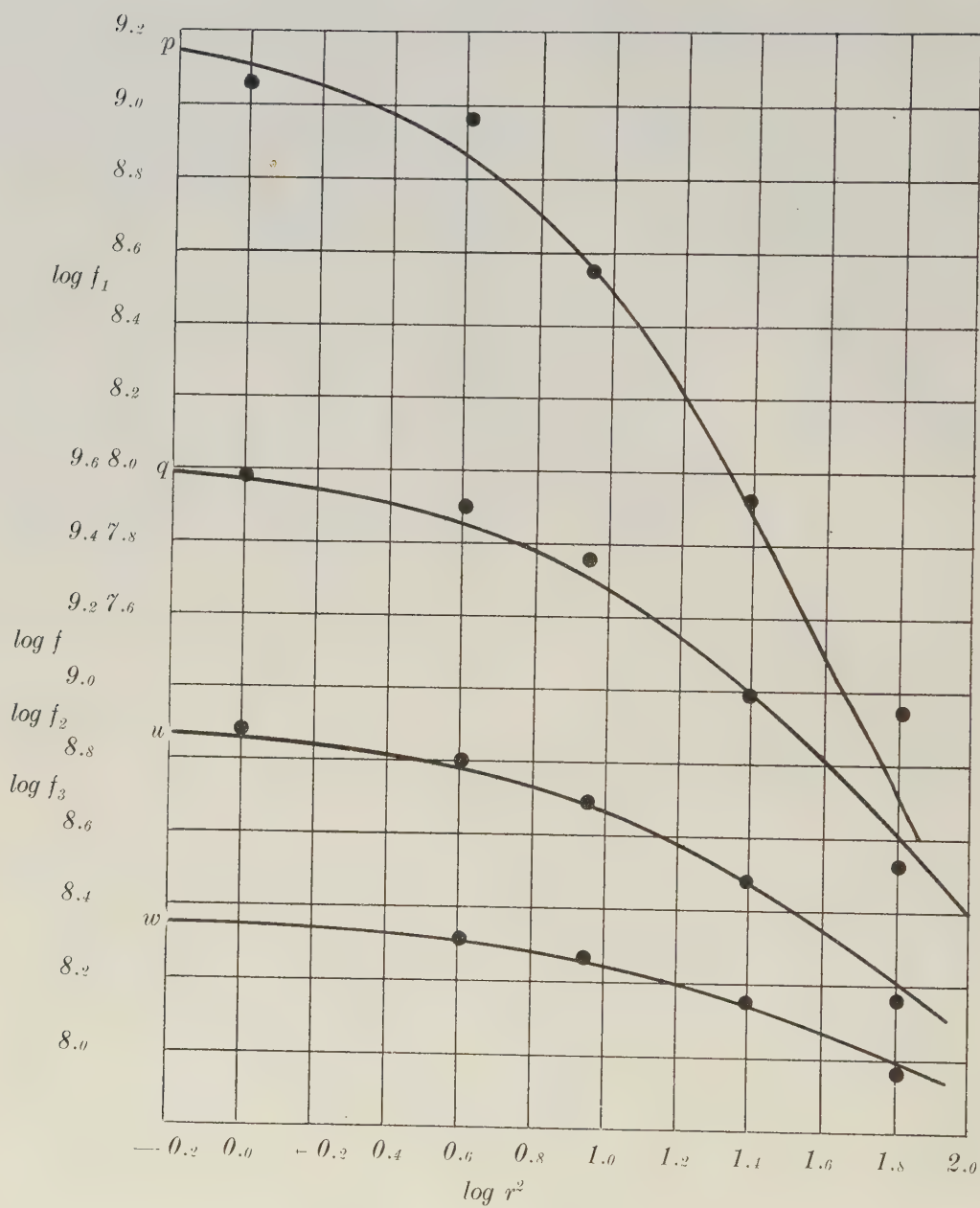


Fig. 1.

where the suffixes *obs* and *cal* designate observed and calculated values respectively. But as ε is a small quantity we have

$$\log f_{cal}(\log \varrho^2 + 0.5 + \varepsilon) = \log f_{cal}(\log \varrho^2 + 0.5) + \varepsilon \frac{d \log f(\log \varrho^2 + 0.5)}{d \log \varrho^2}$$

and ε is determined by the equations

$$\log f_{obs}(\log r^2) - \log f_{cal}(\log \varrho^2 + 0.5) = \varepsilon \frac{d \log f(\log \varrho^2 + 0.5)}{d \log \varrho^2}$$

The values of $\log f_{obs} - \log f_{cal}$ and of $\frac{d \log f(\log \varrho^2 + 0.5)}{d \log \varrho^2}$ are taken from the preliminary diagram.

Solving the equations valid for the stars of group *q* we find on applying the method of least squares

$$\log (-4 \pi m' \kappa) = -0.452.$$

We are now able to compare the observed distributions with those calculated on the supposition that the cluster is in its most probable state. Both sets of values are given in table 3, and are graphically represented in fig. 1, where the curves represent the theoretical distributions and the dots give the observed densities in space $\log f_k$ of stars of the different groups.

TABLE 3.

$\log r^2$	$\log \varrho^2$	<i>p</i>		<i>q</i>		<i>u</i>		<i>w</i>	
		$\log f_{1\ obs}$	$\log f_{1\ cal}$	$\log f_{obs}$	$\log f_{cal}$	$\log f_{2\ obs}$	$\log f_{2\ cal}$	$\log f_{3\ obs}$	$\log f_{3\ cal}$
$-\infty$	$-\infty$	9.233—10	9.210—10	9.604—10	9.613—10	8.901—10	8.888—10		
0.000	-0.452	9.182	9.113	9.581	9.568	8.882	8.858		
0.602	+0.150	8.972	8.859	9.505	9.450	8.802	8.778	8.313—10	8.301—10
0.954	0.502	8.556	8.537	9.367	9.300	8.693	8.678	8.264	8.247
1.398	0.946	7.927	7.874	8.993	8.992	8.476	8.470	8.154	8.136
1.806	1.354	(7.350)	7.026	8.526	8.597	8.161	8.250	7.960	7.994
Number of stars		57		795		682		1203	

Let us first remark that although the value of $\log (-4 \pi m' \kappa)$ has been determined from group *q* the same value would obviously have been obtained from groups *u* and *w*. With regard to the stars of group *p* the agreement would certainly have been a better one for another value of $\log (-4 \pi m' \kappa)$ but these stars are too few to be taken into full consideration. But the constancy of m' for groups of stars of different mass indicates the existence of equipartition of energy within the cluster.

As already observed this numerical calculation is not based on all the observed values, that of $\log f_1(8) = 7.350-10$ being excluded. v. ZEIPPEL and LINDGREN found that the least-square-solution fitted better to the observed values when

$\log f_1(8)$ was excluded. However, when my calculation was finished Professor v. ZEIPPEL told me that Filosofie Licentiat Å. WALLENQUIST, in a paper, as yet unpublished,¹⁾ had applied EDDINGTON's mass-luminosity-relation to the stars of M 37 and without excluding the value of $\log f_1(8)$ had found a perfect agreement for all groups of stars except the g -dwarfs. On the recommendation of Professor v. ZEIPPEL I have also solved a differential equation based on *all* the observed values.

According to v. ZEIPPEL and LINDGREN the value of c_k and μ_k are then

$$\begin{array}{llll} c_1 = 0.164, & c = 0.000, & c_2 = 0.851, & c_3 = 1.501 \\ \mu_1 = 1.762, & \mu = 1.000, & \mu_2 = 0.673, & \mu_3 = 0.360 \end{array}$$

and the differential equation is

$$\frac{d^2 \log f}{dr^2} + \frac{2}{r} \frac{d \log f}{dr} = 4\pi m' \kappa \left(f + 1.208 f^{1.762} + 0.095 f^{0.673} + 0.011 f^{0.360} \right)$$

Starting again with the values

$$\varrho = 0, \quad y = 0, \quad \log f(0) = 9.613 - 10$$

I obtain the values of $\log f_1$, $\log f$, $\log f_2$, and $\log f_3$ given in table 4.

TABLE 4.

ϱ	$\log \varrho^2$	$\log f_1$	$\log f$	$\log f_2$	$\log f_3$
0.00	— ∞	9.154—10	9.613—10	8.888—10	8.360—10
0.16	8.408—10	9.149	9.610	8.887	8.359
0.32	9.010	9.133	9.601	8.880	8.355
0.48	9.362	9.107	9.586	8.870	8.350
0.64	9.612	9.071	9.566	8.857	8.343
0.80	9.806	9.027	9.541	8.840	8.334
1.12	0.098	8.920	9.480	8.799	8.312
1.44	0.317	8.791	9.407	8.750	8.286
1.76	0.491	8.650	9.327	8.696	8.257
2.08	0.636	8.504	9.244	8.640	8.227
2.40	0.760	8.354	9.159	8.583	8.196
2.72	0.869	8.210	9.077	8.528	8.167
3.04	0.966	8.065	8.995	8.473	8.137
3.68	1.132	7.792	8.840	8.368	8.081
4.32	1.271	7.540	8.697	8.272	8.030
5.28	1.445	7.204	8.506	8.144	7.961

¹⁾ The paper is entitled »A research based on the bolometric magnitudes in the cluster Messier 37 (N. G. C. 2099)» and has been accepted for publication in the »Arkiv för matematik, astronomi och fysik utgivet av K. Svenska Vetenskapsakademien». I am indebted to Filosofie Licentiat Å. WALLENQUIST for his kindness in communicating me the results of his research. (Note added to proof: The paper has appeared in Vol. 20 A Nr. 26).

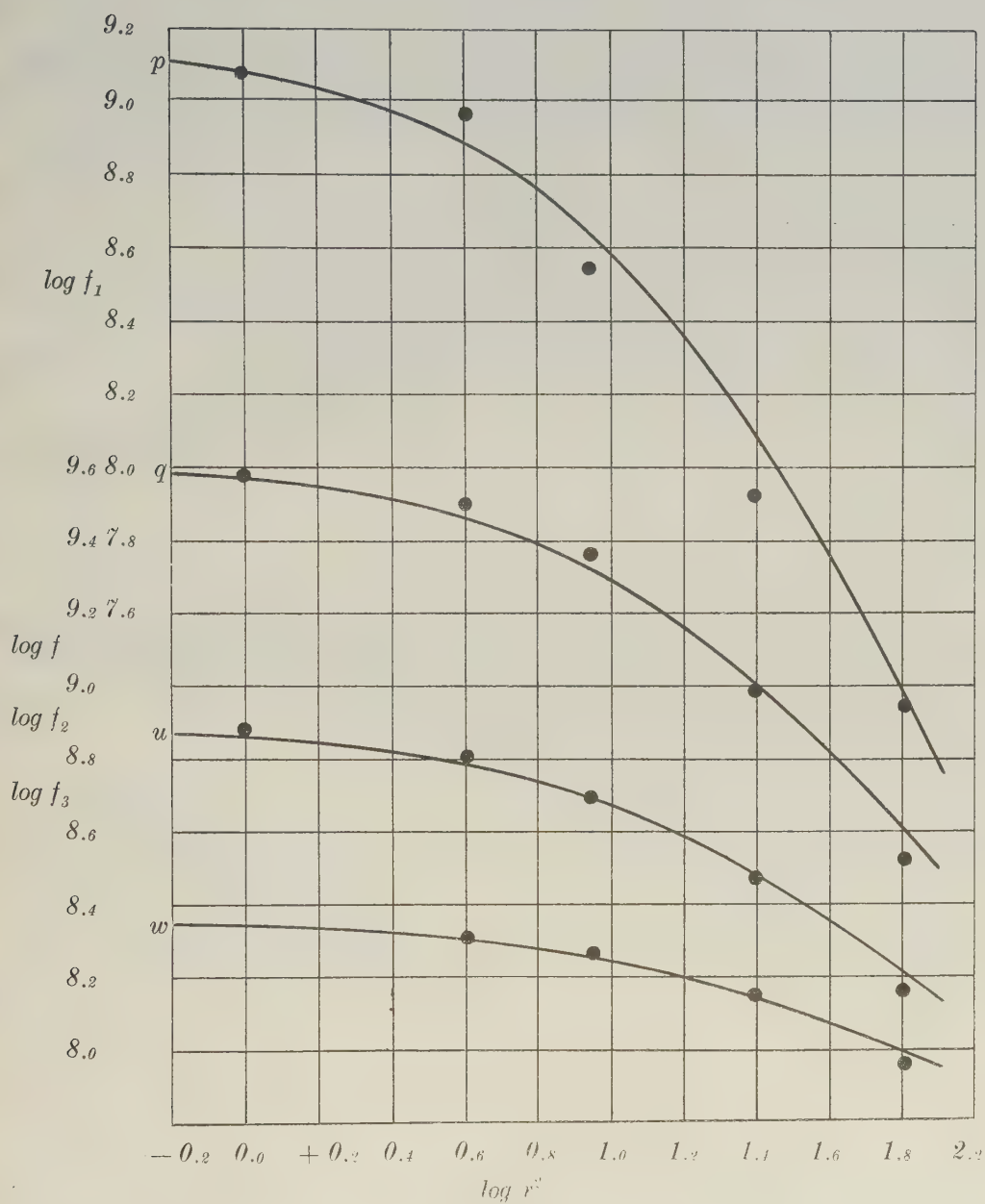


Fig. 2.

Adjusting as before the axis of $\log \varrho^2$ to the axis of $\log r^2$ we again find

$$\log (-4 \pi m' \kappa) = -0.452.$$

The observed distribution (given by dots), and the new theoretical one are graphically represented in fig. 2. The corresponding values are given in table 5.

TABLE 5.

$\log r^2$	$\log \varrho^2$	p		q		u		w	
		$\log f_1 \text{ obs}$	$\log f_1 \text{ cal}$	$\log f_{\text{obs}}$	$\log f_{\text{cal}}$	$\log f_2 \text{ obs}$	$\log f_2 \text{ cal}$	$\log f_3 \text{ obs}$	$\log f_3 \text{ cal}$
$-\infty$	$-\infty$	9.233—10	9.154—10	9.604—10	9.613—10	8.901—10	8.888—10		
0.000	-0.452	9.182	9.082	9.581	9.572	8.882	8.861		
0.602	+0.150	8.972	8.893	9.505	9.465	8.802	8.789	8.313—10	8.306—10
0.954	0.502	8.556	8.640	9.367	9.321	8.693	8.692	8.264	8.255
1.398	0.946	7.927	8.095	8.993	9.012	8.476	8.484	8.154	8.143
1.806	1.354	7.350	7.383	8.526	8.608	8.161	8.212	7.960	7.998

We first observe that the stars represented by $\log f_1 (8)$ are too few to cause any sensible deflection of the curves for q -, u - and w -stars, but the difference existing between the calculated values in table 3 and in table 5 is slightly favouring the last calculation. As for the p -stars there appears to be a decisively closer agreement when the value of $\log f_1 (8)$ is included.

However, all the differences (except one $\log f_1 (8)$), show that the stars in M 37 are clustered in a denser way than is required by the laws of the most probable distribution. This gives rise to the thought that the distribution is not the most probable one but is adiabatic. This fact is, however, just as easily explained by assuming the distribution to be the most probable one and the cluster to contain stars insufficiently luminous to be observed. These hypothetical stars do not change the values of $\log f_k$. The potential, however, cannot be represented any longer by

$$\varphi(r) = \frac{1}{r} \int_0^r 4 \pi r'^2 \sum_{i=1}^L \mu_i f_i(r') dr' + \int_r^R 4 \pi r' \sum_{j=1}^L \mu_j f_j(r') dr'$$

but contains supplementary terms arising from the hypothetical stars. As a consequence the theoretical distribution presents a denser concentration and the discrepancies between the observed and calculated values of $\log f_k$ may thus be accounted for and eliminated.

We may thus be warranted in concluding that the distribution in the open cluster M 37 is in its most probable state.

As observed in the Introduction, it has not been possible to advance any rational explanation justifying the theory that a globular cluster is adiabatically distributed. I think the observed fact, $n = 5$, may be accepted as an indication that the clusters are, in reality, in their most probable distribution (isothermal) and contain stars of different masses, the largest masses only being observed from the earth. I hope that further observations will make a closer examination of this suggestion possible.

In conclusion I wish to express my heartfelt thanks to Professor H. v. ZEIPPEL for the valuable interest he has shown in my work.
